

# Estimates on the tail behaviour of Gaussian polynomials. The discussion of a result of Latała.

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*Summary:* In this paper a result of Latała about the tail behaviour of Gaussian polynomials will be discussed. Latała proved an interesting result about this problem in paper [2]. But his proof applied an incorrect statement at a crucial point. Hence the question may arise whether the main result of paper [2] is valid. The goal of this paper is to settle this problem by presenting such a proof where the application of the erroneous statement is avoided. I discuss the proofs in detail even at the price of a longer text and try to give such an explanation that reveals the ideas behind them better than the original paper.

## 1. Introduction. Formulation of the main results.

In this paper the following problem studied in Latała's paper [2] will be revisited.

Let us have a multilinear form

$$\begin{aligned} A(u_1, \dots, u_d) &= A(d)(u_1, \dots, u_d) \\ &= \sum_{(i_1, \dots, i_d): 1 \leq i_j \leq n_j, 1 \leq j \leq d} a(i_1, \dots, i_d) u_1(i_1) \cdots u_d(i_d) \end{aligned} \quad (1.1)$$

of order  $d$  in the space of vectors  $(u_1, \dots, u_d)$  where  $u_j = (u_j(1), \dots, u_j(n_j)) \in R^{n_j}$ , and  $R^{n_j}$  is the Euclidean space with some prescribed dimension  $n_j$ ,  $1 \leq j \leq d$ . The set of real numbers  $A(d) = A(d|n_1, \dots, n_d) = \{a(i_1, \dots, i_d), 1 \leq i_j \leq n_j, 1 \leq j \leq d\}$  are also prescribed in this formula.

Beside this, let us also have  $d$  independent standard Gaussian random vectors  $G_j = (g_j(1), \dots, g_j(n_j))$  of dimension  $n_j$ ,  $1 \leq j \leq d$ , and define with the help of the multilinear form (1.1) and these Gaussian random vectors the Gaussian random polynomial

$$Y(A) = Y(A(d)) = \sum_{(i_1, \dots, i_d): 1 \leq i_j \leq n_j, 1 \leq j \leq d} a(i_1, \dots, i_d) g_1(i_1) \cdots g_d(i_d) \quad (1.2)$$

of order  $d$ . We want to give a good estimate on the tail distribution  $P(|Y(A)| > x)$  for all  $x > 0$  under appropriate conditions on the multilinear form  $A(d)(\cdot)$  defined in (1.1). Naturally, it belongs to the problem to find the right conditions under which useful results can be proved.

Some estimates can be proved about the tail distribution of Gaussian polynomials and so-called degenerate  $U$ -statistics under the condition that their variance is bounded by a known constant, (see [5]), and these results are in a sense sharp. On the other hand,

they can be improved if we have some additional useful information about the behaviour of the multi-linear form (1.1). Latała proved an interesting result in this direction in paper [2]. He found the right conditions under which a good estimate can be given about the tail-distribution  $P(|Y(A)| > x)$ . Similar questions can be also asked about degenerate  $U$ -statistics, and Adamczak proved in [1] some results in this direction. But the essential step in the study of such problems is to find the proof (and formulation) of the right estimates for the tail distribution of Gaussian polynomials. The adaptation of such results to  $U$ -statistics is rather a technical problem.

Hence I restrict my attention to Latała's work. I discuss its proof and present a version of it, because I found an error in paper [2] that caused serious problems for me. For a long time I have even doubted the validity of the main result in [2]. My problems were related to the proof of Theorem 3 in [2]. It was based on a backward induction procedure with respect to a parameter  $l$ . The induction steps when we turn from  $l + 1$  to  $l$  were explained for all parameters  $l \geq 1$ . But the final step when we turn from  $l = 1$  to  $l = 0$  was not considered in the proof. Moreover, the arguments of the paper do not work in this case, and as a consequence the proof of Theorem 3 is invalid. At the end of Section 8 I discuss this problem in a remark in more detail.

The above mentioned error seems to be crucial. I believe that not only the proof but even the formulation of Theorem 3 is erroneous. Since the proof of the main result of paper [2] depends heavily on this theorem the question arises whether this result holds. It demanded much work from me to clarify this problem. Finally I found a correct proof of the main result of paper [2] which does not apply Theorem 3 of [2]. I present it in this paper. Beside this I also try to explain its main ideas.

To formulate Latała's result I introduce some notations. Let us define the linear functional

$$A(v) = A(d, v) = \sum_{(i_1, \dots, i_d): 1 \leq i_j \leq n_j, 1 \leq j \leq d} a(i_1, \dots, i_d) v(i_1, \dots, i_d) \quad (1.3)$$

in the space of all real valued functions  $v(i_1, \dots, i_d)$  defined on the  $n$ -tuples  $(i_1, \dots, i_d)$ ,  $1 \leq i_j \leq n_j$ ,  $1 \leq j \leq d$ , where the coefficients  $a(i_1, \dots, i_d)$  agree with those given in (1.1)

Let us also introduce the class  $\mathcal{P} = \mathcal{P}_d$  consisting of all partitions of the set  $\{1, \dots, d\}$ . We shall define a class of finite series of functions with the help of these partitions, and the conditions of Latała's result will be formulated with their help. To avoid some repetitions in further discussions I define these quantities in a slightly more general form.

Let us have a finite subset  $K \subset \{1, 2, \dots\}$  of the positive integers together with a function  $b_K(i_j, j \in K)$ ,  $1 \leq i_j \leq n_j$ ,  $j \in K$ , and the numbers  $n_j$ ,  $j \in K$ , which tell what values the arguments of the function  $B_K(i_j, j \in K, 1 \leq j \leq n_j, j \in K)$  can take. We define with their help, similarly to the quantity  $A(v)$ , the linear functional

$$B_K(v) = \sum_{(i_j, j \in K): 1 \leq i_j \leq n_j, j \in K} b_K(i_j, 1 \leq i_j \leq n_j, j \in K) v(i_j, 1 \leq i_j \leq n_j, j \in K) \quad (1.4)$$

on the space of functions  $v(i_j, 1 \leq i_j \leq n_j, j \in K)$ .

Let  $\mathcal{P}(K)$  denote the set of all partitions of the set  $K$ , and given a partition  $P = \{A_1, \dots, A_s\} \in \mathcal{P}(K)$  of  $s$  elements together with the positive integers  $n_j, j \in K$ , appearing in the definition of the sets  $P(K)$  let us define with their help the following set  $\mathcal{G}_P$  of sequences of functions  $(v_1, v_2, \dots, v_s)$ :

$$\mathcal{G}_P = \left\{ (v_1(i_j, 1 \leq i_j \leq n_j, j \in A_1), \dots, v_s(i_j, 1 \leq i_j \leq n_j, j \in A_s)) : \sum_{(i_1, \dots, i_j): 1 \leq i_j \leq n_j, j \in A_r} v_r^2(i_j, j \in A_r) \leq 1 \text{ for all } 1 \leq r \leq s \right\} \quad (1.5)$$

if  $P = \{A_1, \dots, A_s\} \in \mathcal{P}(K)$ . Let us have a linear functional  $B_K(v)$  of the form (1.4) together with the coefficients  $b_K(\cdot)$  taking part in its definition. Then we define with the help of the class of functions  $\mathcal{G}_P$  defined in (1.5) the following quantity  $V(P, B_K)$  for all partitions  $P \in \mathcal{P}(K)$ .

$$V(P, B_K) = V(P, b_K(\cdot)) = \sup_{(v_1, \dots, v_s) \in \mathcal{G}_P} \sum b_K(i_j, 1 \leq i_j \leq n_j, j \in K) \prod_{1 \leq r \leq s} v_r(i_j, 1 \leq i_j \leq n_j, j \in A_r). \quad (1.6)$$

for a partition  $P = \{A_1, \dots, A_s\} \in \mathcal{P}(K)$ . In this formula the same coefficients  $b_K(i_j, 1 \leq i_j \leq n_j, j \in K)$  appear as in (1.4).

Given a partition  $P = \{A_1, \dots, A_s\} \in \mathcal{P}(K)$  let  $|P| = s$  denote its cardinality. In the remaining part of this section I restrict my attention to partitions  $P \in \mathcal{P}_d$  of the set  $\{1, \dots, d\}$  and to the case when the linear functional  $A(v)$  defined in formula (1.3) is considered. In this case the quantity introduced in (1.6) will be denoted as  $V(P, A) = V(P, (a(\cdot)))$ . Let us define with its help the numbers

$$\alpha_s = \alpha_s(A) = \sup_{P: P \in \mathcal{P}_d, |P|=s} V(P, A) \text{ for all } 1 \leq s \leq d. \quad (1.7)$$

The main result of Latała we discuss in this paper can be formulated with the help of the quantities  $\alpha_s, 1 \leq s \leq d$ , introduced in (1.7). It states the following inequalities.

**Theorem 1.** *The moments of the Gaussian random polynomial  $Y(A(d))$  defined in formula (1.2) satisfy the inequality*

$$E(Y(A(d)))^{2M} \leq \left( C(d) \max_{1 \leq s \leq d} (M^{s/2} \alpha_s) \right)^{2M} \quad (1.8)$$

for all  $d \geq 2$  and  $M = 1, 2, \dots$  with the quantities  $\alpha_s$  defined in (1.7) and a constant  $C(d)$  depending only on the order  $d$  of the Gaussian polynomial  $Y(A(d))$ . As a consequence,

$$P(|Y(A(d))| > x) \leq C(d) \exp \left\{ -\frac{1}{C(d)} \min_{1 \leq s \leq d} \left( \frac{x}{\alpha_s} \right)^{2/s} \right\} \quad (1.9)$$

for all  $d \geq 2$  and  $x > 0$  with some constant  $C(d)$  depending only on  $d$ .

*Remark 1.* Latała's paper also contains a similar lower bound for the moments and probabilities in (1.8) and (1.9). These bounds state that the estimates in this formulas are essentially sharp, only the value of the parameter  $C(d)$  can be improved in them. The proof of these lower bounds is considerably simpler. Since their proof in [2] is correct, I shall omit their discussion.

*Remark 2.* In the subsequent estimations some constants  $C$ ,  $C_1$ ,  $C(d)$  etc. will appear in different formulas. The same letter may denote different constants in different formulas. It will be important that these constants are universal, depending at least of the order  $d$  of the Gaussian polynomial we are considering. There will be some places in our discussion where the relation between constants in different formulas have to be investigated. The necessary considerations will be taken at these points.

*Remark 3.* The dimension  $n_j$  of the Euclidean spaces  $R^{n_j}$  where the appropriate vectors take their values plays no role in our considerations. It is exploited in some arguments that they are finite, but their value will be not important for us. At several points where it makes no problem I shall omit the parameters  $n_j$  from the formulas. By means of some limiting procedure one can get results in infinite dimensional spaces, but this will be not done here.

I formulate a formally weaker version of Theorem 1 in the following Theorem 1A. But actually, as I shall show these two results are equivalent. Since Theorem 1A is technically simpler, this result will be proved.

**Theorem 1A.** *Let the Gaussian polynomial  $Y(A(d))$ ,  $d \geq 2$ , defined in (1.2) be such that the expressions  $\alpha_s$ ,  $1 \leq s \leq d$ , defined in (1.7) satisfy the inequality*

$$\alpha_s = \alpha_s(A) \leq M^{-(s-1)/2} \quad \text{for all } 1 \leq s \leq d \quad (1.10)$$

*with some positive integer  $M$ . Then*

$$EY(A(d))^{2M} \leq C(d)^M M^M \quad (1.11)$$

*with a constant  $C(d) > 0$  depending only on the order  $d$  of the Gaussian polynomial  $Y(A(d))$ .*

Theorem 1A states that if a Gaussian polynomial  $Y(A(d))$  satisfies condition (1.10) then its  $2M$ -th moment satisfies such an estimate as the  $2M$ -th moment of a standard normal random variables multiplied by a constant.

*The deduction of Theorem 1 from Theorem 1A.* Let us consider the random variable  $Y(A(d))$  and the number  $2M$  which is the moment we consider in formula (1.8). Let us define with their help the constant  $D(M) = \max_{1 \leq s \leq d} (M^{(s-1)/2} \alpha_s)$  and introduce the Gaussian polynomial  $D(M)^{-1}Y(A(d))$  defined in formula (1.2) with coefficients

$D(M)^{-1}a(i_1, \dots, i_d)$ . This polynomial satisfies relation (1.10), hence by Theorem 1A relation (1.11) also holds for it. This means that  $EY(A(d))^{2M} \leq (C(d)D(M)^2M)^M$  which is equivalent to relation (1.8) in Theorem 1.

Relation (1.9) follows from relation (1.8) in the standard way. By the Markov inequality  $P(|Y(A(d))| \geq x) \leq x^{-2M} EY(A(d))^{2M}$  for arbitrary  $M = 1, 2, \dots$ . Choose

$M = \left\lceil \min_{1 \leq s \leq d} \frac{1}{KC(d)} \frac{x}{\alpha_s} \right\rceil^{2/s}$  if  $x \geq KC(d) \min_{1 \leq s \leq d} \alpha_s$ , where  $[\cdot]$  denotes integer part,  $C(d)$  is the same constant which appears in (1.8), and  $K = K(d)$  is a sufficiently large constant depending only on  $d$ . In this case we get from relation (1.8) that  $P(|Y(A(d))| \geq x) \leq e^{-M}$  which implies relation (1.9) with the constant  $K^2C(d)^2$  if  $x \geq KC(d) \min_{1 \leq s \leq d} \alpha_s$ .

On the other hand, if  $x \leq KC(d) \min_{1 \leq s \leq d} \alpha_s$ , and the constant  $K$  was chosen sufficiently large, then the right-hand side of relation (1.9) (with the previously chosen constant  $K^2C(d)^2$  as the number ‘ $C(d)$ ’ in (1.9)) is larger than 1. Hence relation (1.9) holds also in this case.

This paper consists of eight sections and an Appendix. In Section 2 the proof of Theorem 1A is reduced to a result called the Basic estimate by means of a conditioning argument. In Section 3 this Basic estimate is proved in the special case  $d = 2$ . In Section 4 a result of paper [2] is recalled about the estimation of the cardinality of an appropriate  $\varepsilon$ -net in a metric space with some nice properties. In Section 5 a result called the Main inequality is presented, and it is shown that the Basic estimate follows from it. In Section 6 two results, Lemma 6.1 and Lemma 6.2 are formulated. They provide a good partition of certain sets of functions which play crucial role in the proof of the Main inequality. The proof of these lemmas is based on some estimates formulated in Lemma 6.3. Lemma 6.3 together with its proof is also given in Section 6. Lemmas 6.1 and 6.2 are proved in Section 7. Finally the Main inequality is proved in Section 8 by means of the results in Section 6. Since in Section 4 I apply a terminology essentially different from that of [2] I found better not to refer to the original proofs of the results presented here, but to describe them instead. This is done in the Appendix. In such a way I wanted to make this paper self-contained.

The proofs of this paper apply several ideas of Paper [2]. But since the notation and the formulation of the results in these two works are very different, and the main ideas in [2] are presented in a rather hidden way I only explain which results of these two papers correspond to each other.

## 2. The application of a conditioning argument.

In this section a conditioning argument is applied to reduce the proof of Theorem 1A to the verification of a result called the Basic estimate.

To carry out this conditioning argument let us define the Gaussian random vector

$$Y_d(u) = Y_d(u, A) = \sum_{(i_1, \dots, i_d): 1 \leq i_j \leq n_j, 1 \leq j \leq d} a(i_1, \dots, i_d) u_1(i_1) \dots u_{d-1}(i_{d-1}) g_d(i_d) \quad (2.1)$$

for all vectors  $u = (u_1, \dots, u_{d-1})$ ,  $u_j = (u_j(1), \dots, u_j(n_j))$ ,  $1 \leq j \leq d-1$ , and a standard Gaussian vector  $G_d = (g_d(i_1), \dots, g_d(n_d))$ . The coefficients  $a(i_1, \dots, i_d)$  in formulas (1.1) and (2.1) are the same. Actually in formula (2.1) we took the multilinear form (1.1) and replaced the vector  $u_d$  by the standard normal random vector  $G_d$  in it.

We want to estimate the moments of the random variables  $Y(A(d))$  introduced in (1.2). This can be done by means of the following conditioning argument.

$$\begin{aligned} E(Y(A(d))^{2M} | g_d(1) = u_d(1), \dots, g_d(n_d) = u_d(n_d)) \\ = E \left( \sum_{(i_1, \dots, i_d): 1 \leq i_j \leq n_j, 1 \leq j \leq d} a(i_1, \dots, i_d) g_1(i_1) \dots g_{d-1}(i_{d-1}) u_d(i_d) \right)^{2M}, \end{aligned}$$

hence

$$EY(A(d))^{2M} = EY(A(d), M, G_d) \quad (2.2)$$

with

$$\begin{aligned} Y(A(d), M, u_d) \\ = E \left[ \sum_{i_d=1}^{n_d} \left( \sum_{(i_1, \dots, i_{d-1}): 1 \leq i_j \leq n_j, 1 \leq j \leq d-1} a(i_1, \dots, i_d) g_1(i_1) \dots g_{d-1}(i_{d-1}) \right) u_d(i_d) \right]^{2M}, \end{aligned}$$

or in an equivalent form

$$\begin{aligned} Y(A(d), M, u_d) \\ = E \left( \sum_{(i_1, \dots, i_{d-1}): 1 \leq i_j \leq n_j, 1 \leq j \leq d-1} b_{u_d}(i_1, \dots, i_{d-1}) g_1(i_1) \dots g_{d-1}(i_{d-1}) \right)^{2M}, \end{aligned} \quad (2.3)$$

with

$$b_{u_d}(i_1, \dots, i_{d-1}) = \sum_{i_d=1}^{n_d} a(i_1, \dots, i_d) u_d(i_d), \quad (2.4)$$

where  $u_d = (u_d(1), \dots, u_d(n_d))$  is an arbitrary vector in  $R^{n_d}$ .

Next I formulate a result called the Basic estimate. Its proof will be the main subject of the subsequent sections. Here I prove that Theorem 1A follows from it. To formulate the Basic estimate first I introduce the following quantity.

$$Z_d = Z_d(A) = \sup_{u=(u_1, \dots, u_{d-1}): u_j \in B^{n_j}, 1 \leq j \leq d-1} Y_d(u), \quad (2.5)$$

where the (Gaussian) random variables  $Y_d(u)$  were defined in (2.1). Here and in the subsequent part of the paper  $B^n$  denotes the unit ball in the Euclidean space  $R^n$  with the usual Euclidean norm, i.e.  $B^n = \{(u(1), \dots, u(n)): \sum_{j=1}^n u(j)^2 \leq 1\}$ . It will be shown with the help of the previous calculations that Theorem 1A follows from the following result.

**Basic estimate.** *If the linear form  $A(v)$ ,  $d \geq 2$ , introduced in (1.3) is such that the quantities  $\alpha_s$  defined in (1.7) satisfy the condition (1.10) with some positive integer  $M$ , i.e.  $\alpha_s = \alpha_s(A) \leq M^{-(s-1)/2}$  for all  $1 \leq s \leq d$ , then the estimate*

$$EZ_d^{2M} = EZ_d(A)^{2M} \leq C^M M^{-(d-2)M} \quad (2.6)$$

*holds with a constant  $C = C(d)$  depending only on  $d$ .*

*Remark.* The above formulated Basic estimate is closely related to Theorem 2 in [2]. The main difference between them is that Theorem 2 in [2] gives an estimate only for the expected value  $EZ_d(A)$  of  $Z_d(A)$  and not for its higher moments. Thus our result is, — at least formally, — sharper. But actually estimate (2.6) follows from the result of [2] and an important concentration inequality of Ledoux about the supremum of Gaussian random variables. This result will be recalled in Section 3. The reason for the present formulation of the Basic estimate was that I wanted to show that the so-called chaining argument applied in its proof also supplies the estimate (2.6) for  $d \geq 3$ , i.e. we do not need Ledoux's inequality in this case. Surprisingly, we need it just in the simplest case  $d = 2$ , when the proof is given by means of a simple and natural direct calculation instead of the chaining argument.

We shall estimate  $EY(A(d))^{2M}$  with the help of relations (2.2) and (2.3) by induction with respect to  $d$  for all  $d \geq 2$ . Let us first consider the case  $d = 2$ .

If the linear form  $A(2)(u_1, u_2)$  in (1.1) (with  $d = 2$ ) is defined with the help of a set of numbers  $\{a(i, j) \mid 1 \leq i \leq n_1, 1 \leq j \leq n_2\}$ , then we can write

$$\begin{aligned} Y(A(2), M, u_2) &= E \left[ \sum_{i=1}^{n_1} \left( \sum_{j=1}^{n_2} a(i, j) u_2(j) \right) g_1(i) \right]^{2M} \\ &= 1 \cdot 3 \cdot \dots \cdot (2M - 1) \left( E \left[ \sum_{i=1}^{n_1} \left( \sum_{j=1}^{n_2} a(i, j) u_2(j) \right) g_1(i) \right]^2 \right)^M \end{aligned}$$

$$\begin{aligned}
&= 1 \cdot 3 \cdot \dots \cdot (2M-1) \left( \sum_{i=1}^{n_1} \left( \sum_{j=1}^{n_2} a(i,j) u_2(j) \right)^2 \right)^M \\
&= 1 \cdot 3 \cdot \dots \cdot (2M-1) \left( \sup_{u_1=(u_1(1), \dots, u_1(n_1)): u_1 \in B^{n_1}} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} a(i,j) u_1(i) u_2(j) \right)^{2M},
\end{aligned} \tag{2.7}$$

where  $u_2 = (u_2(1), \dots, u_2(n_2)) \in R^{n_2}$ ,  $u_1 = (u_1(1), \dots, u_1(n_1)) \in B^{n_1}$ , and  $B^{n_1}$  denotes the unit ball of the Euclidean space  $R^{n_1}$ , i.e. we demand that  $\sum_{i=1}^{n_1} u_1(i)^2 \leq 1$ . By relations (2.2), (2.7), the definition of the quantity  $Z_d(A)$  and the Basic estimate

$$EY_2(A(2))^{2M} \leq (2M)^M EZ_2(A(2))^{2M} \leq CM^M$$

if  $\alpha_1(A) \leq 1$  and  $\alpha_2(A) \leq M^{-1/2}$ , i.e. if the conditions of the Basic estimate hold for  $d = 2$ . Thus we have proved Theorem 1A with the help of the Basic estimate in the case  $d = 2$ .

In the case  $d \geq 3$  Theorem 1A will be proved by means of induction. During this induction procedure we assume that Theorem 1A holds for  $2 \leq d' \leq d-1$ , and the Basic estimate holds for  $2 \leq d' \leq d$ .

First the expression  $Y(A(d), M, u_d)$  will be estimated. This expression, defined in (2.3) is the  $2M$ -th moment of a Gaussian polynomial of order  $d-1$ . It is defined similarly to  $Y_d(u)$  introduced in formula (2.1) only with the coefficients  $b_{u_d}(i_1, \dots, i_{d-1})$  introduced in (2.4) instead of  $a(i_1, \dots, i_d)$ . Hence, as we shall show, they satisfy the following inequality.

$$\begin{aligned}
Y(A(d), M, u_d) &\leq \max_{P \in \mathcal{P}_{d-1}} \left( V(P, B_{u_d})^2 M^{|P|-1} \right)^M (CM)^M \\
&\leq C^M \sum_{P \in \mathcal{P}_{d-1}} V(P, B_{u_d})^{2M} M^{|P|M}
\end{aligned} \tag{2.8}$$

with some constant  $C = C(d)$ , where  $V(P, B_{u_d})$  was defined in (1.6) for partitions  $P \in \mathcal{P}_{d-1}$  i.e.  $K = \{1, \dots, d-1\}$ , and the numbers  $b_{u_d}(i_1, \dots, i_{d-1})$  introduced in (2.4) play the role of the coefficients  $b_K(\cdot)$  in formulas (1.4) and (1.6).

Indeed, the expression  $\frac{Y(A(d), M, u_d)}{\max_{P \in \mathcal{P}_{d-1}} (V(P, B_{u_d}) M^{|P|-1/2})^{2M}}$  equals the  $2M$ -th moment of such a Gaussian polynomial which satisfies the conditions of Theorem 1A with parameter  $d-1$ . Hence Theorem 1A with parameter  $d-1$  (which holds by our induction hypothesis) implies the first inequality in (2.8). The second inequality of (2.8) is obvious.

By relations (2.2) and (2.8)

$$EY(A(d))^{2M} \leq C^M \sum_{P \in \mathcal{P}_{d-1}} EV(P, B_{G_d})^{2M} M^{|P|M},$$



where  $V(P, B_{G_d})$  is the random variable we get by replacing the vector  $u_d$  by the random vector  $G_d = (g_d(1), \dots, g_d(n_d))$  in the expression  $V(P, B_{u_d})$ . Hence to complete the proof of the Theorem 1A it is enough to show that under the conditions of Theorem 1A

$$EV(P, B_{G_d})^{2M} \leq C^M M^{-(|P|-1)M} \quad \text{for all } P \in \mathcal{P}(\{1, \dots, d-1\}) \quad (2.9)$$

with a constant  $C = C(d)$ . This result can be proved with the help of the Basic estimate.

To prove formula (2.9) take a partition  $P = \{A_1, \dots, A_s\} \in \mathcal{P}_{d-1}$  with  $|P| = s$  elements. With such a choice

$$V(P, B_{G_d}) = \sup_{(v_1, \dots, v_s) \in \mathcal{G}_P} \sum_{(i_1, \dots, i_d)} a(i_1, \dots, i_d) \prod_{r=1}^s v_r(i_j, j \in A_r) g_d(i_d). \quad (2.10)$$

In formula (2.10) the class of functions  $\mathcal{G}_P$  where the supremum is taken is defined in (1.5) with the partition  $P$  we have fixed, and  $(g_d(1), \dots, g_d(n_d))$  is an  $n_d$  dimensional standard normal vector. The  $2M$ -th moment of the right-hand side expression in (2.10) can be bounded by means of the Basic estimate with  $s+1 = |P|+1 \leq d$  parameters (i.e. the number  $|P|+1$  takes the role of the parameter  $d$  in this case) if the vectors  $(i_j, j \in A_r)$ ,  $A_r \in P$ , are considered as one variable for all  $1 \leq r \leq s$ . The condition of the Basic estimate formulated in (1.10) holds with such a choice, and we get inequality (2.9) in such a way.

We have reduced the problem we want to solve to the proof of an inequality formulated in the Basic estimate, where certain moments of a supremum  $\sup_{u \in B^{n_1} \times \dots \times B^{n_{d-1}}} Y_d(u)$  of Gaussian random variables are bounded. The random variables  $Y_d(u)$  in this formula were defined in (2.1), and  $B^n$  denotes the unit ball in  $R^n$ . In the study of such problems it is worth introducing the metric  $\rho(u, v) = [E(Y_d(u) - Y_d(v))^2]^{1/2}$  on the parameter set of the random variables we are considering. This led to the definition of the following pseudometric  $\rho_\alpha$  in the space  $R^{n_1} \times \dots \times R^{n_{d-1}}$ .

$$\begin{aligned} \rho_\alpha(u, v) &= \rho_\alpha((u_1, \dots, u_{d-1}), (v_1, \dots, v_{d-1})) \\ &= [E(Y_d(u) - Y_d(v))^2]^{1/2} = E \left( E \left[ \sum_{1 \leq i_j \leq n_j, 1 \leq j \leq d} a(i_1, \dots, i_d) \right. \right. \\ &\quad \left. \left. (u_1(i_1) \cdots u_{d-1}(i_{d-1}) - v_1(i_1) \cdots v_{d-1}(i_{d-1})) g(i_d) \right]^2 \right)^{1/2} \\ &= \left( \sum_{1 \leq i_d \leq n_d} \left[ \sum_{1 \leq i_j \leq n_j, 1 \leq j \leq d-1} a(i_1, \dots, i_d) \right. \right. \\ &\quad \left. \left. (u_1(i_1) \cdots u_{d-1}(i_{d-1}) - v_1(i_1) \cdots v_{d-1}(i_{d-1})) \right]^2 \right)^{1/2} \end{aligned} \quad (2.11)$$

for all pairs of vectors  $u = (u_1, \dots, u_{d-1})$  and  $v = (v_1, \dots, v_{d-1})$ ,  $u_j \in R^{n_j}$ ,  $v_j \in R^{n_j}$ ,  $1 \leq j \leq d-1$ .

It is useful to give a different characterization of the above introduced metric  $\rho_\alpha$ . For this goal let us define the pseudonorm  $\alpha$

$$\begin{aligned}\alpha(v) &= \alpha_d(v) = \alpha_d(v(i_1, \dots, i_{d-1})) \\ &= \left[ \sum_{1 \leq i_d \leq n_d} \left( \sum_{1 \leq i_j \leq n_j, 1 \leq j \leq d-1} a(i_1, \dots, i_d) v(i_1, \dots, i_{d-1}) \right)^2 \right]^{1/2}\end{aligned}\quad (2.12)$$

in the linear space of the functions  $v = v(i_1, \dots, i_{d-1})$ ,  $1 \leq i_j \leq n_j$ ,  $1 \leq j \leq d-1$ . Clearly,

$$\rho_\alpha((u_1, \dots, u_{d-1}), (v_1, \dots, v_{d-1})) = \alpha_d(u_1 \otimes \dots \otimes u_{d-1} - v_1 \otimes \dots \otimes v_{d-1}) \quad (2.13)$$

where the function  $u_1 \otimes \dots \otimes u_{d-1}$  with arguments  $(i_1, \dots, i_{d-1})$ ,  $1 \leq i_j \leq n_j$  for all  $1 \leq j \leq d-1$  is defined as  $u_1 \otimes \dots \otimes u_{d-1}(i_1, \dots, i_{d-1}) = u_1(i_1) \dots u_{d-1}(i_{d-1})$ , and  $v_1 \otimes \dots \otimes v_{d-1}$  is defined similarly.

The above representation of the metric  $\rho_\alpha$  turned out to be useful. In the study of the Basic estimate we have to find a good  $\varepsilon$ -net for certain subsets of  $B^{n_1} \times \dots \times B^{n_{d-1}}$  with respect to the metric  $\rho_\alpha$  for small  $\varepsilon > 0$ . The representation of the metric  $\rho_\alpha$  by formulas (2.12) and (2.13) may help in finding good  $\varepsilon$ -nets. This question will be discussed in detail in the subsequent sections. But before doing it I prove the Basic estimate together with some related results we need in our discussion in the special case  $d = 2$ . This case is considered separately, because the formulation of the results and their proof for  $d = 2$  are slightly different from those in the general case.

### 3. The proof for Gaussian polynomials of order 2.

In this section the Basic estimate will be proved for Gaussian polynomials of order  $d = 2$ . It will be proved as the consequence of a more general result called the Main inequality in the case  $d = 2$ . A result called the Main inequality will be formulated in Section 5 for all dimensions  $d \geq 3$ . The crucial point in the proof of Theorem 1A is the verification of this result. The Main inequality in the case  $d = 2$  formulated in this section can be considered as a version of this result. But there are some differences between their formulation, and they must be considered separately. The Basic estimate for  $d = 2$  could have been proved directly. I prove it with the help of the Main inequality in the case  $d = 2$ , because the latter result is also needed in the discussion of the case  $d \geq 3$ . To formulate it I introduce some notations.

We shall work with some expressions  $A(v)$  and  $Y_2$  which are the quantities defined in (1.3) and (2.1) in the special case  $d = 2$ . Let us write them down in more detail.

These terms depend on a set of numbers  $A = A(2) = \{a(i, j), 1 \leq i \leq n_1, 1 \leq j \leq n_2\}$ . The first of them is the linear functional

$$A(v) = A(2, v) = \sum_{i,j} a(i, j) v(i, j)$$

in the space of all functions  $v(i, j)$  with arguments  $1 \leq i \leq n_1$ ,  $1 \leq j \leq n_2$ . This is the expression (1.3) in the case  $d = 2$ . The expression (2.1) can be written as

$$Y(u) = Y_2(u) = \sum_{i,j} a(i, j)u(i)g_2(j),$$

with  $u = (u(1), \dots, u(n_1))$ , where  $(g_2(1), \dots, g_2(n_2))$  is a standard normal random vector.

Let us observe that in the case  $d = 2$  the quantity  $\alpha_1(A)$  defined in (1.7) can be written as

$$\alpha_1(A) = \sup_{v(i,j): \sum_{i,j} v(i,j)^2 \leq 1} \sum_{i,j} a(i, j)v(i, j) = \left( \sum_{i,j} a(i, j)^2 \right)^{1/2}. \quad (3.1)$$

Let us also introduce the function

$$\alpha_2(u) = \left[ \sum_j \left( \sum_i a(i, j)u(i) \right)^2 \right]^{1/2} = \sup_{v=(v(1), \dots, v(n_2)): \sum_j v(j)^2 \leq 1} \sum_{i,j} a(i, j)u(i)v(j)$$

for all vectors  $u = (u(1), \dots, u(n_1)) \in R^{n_1}$ .

Let us fix some positive integer  $M$ , and define for all  $N \geq 0$  the following subset  $U_N = U_N(M)$  of  $R^{n_1}$ .

$$U_N = U_N(M) = \{u = (u(1), \dots, u(n_1)): u \in B^{n_1}, \text{ and } \alpha_2(u) \leq 2^{-N} M^{-1/2}\}. \quad (3.2)$$

I formulate with the help of the above notations the following result.

**The Main inequality in the case  $d = 2$ .** *Let  $\alpha_1(A) \leq 1$ . Then the inequality*

$$E \left[ \sup_{u: u \in U_N} Y(u) \right]^{2^{2(N+A)}M} \leq (C \cdot 2^A)^{2^{2(N+A)}M} \quad (3.3)$$

*holds with the sets  $U_N$  defined in (3.2) for all integers  $N \geq 0$ ,  $M \geq 1$  and  $A \geq 1$  with  $C = 2$ .*

*Proof of the Main inequality in the case  $d = 2$ .* This result will be proved with the help of the concentration inequality of Ledoux about the supremum of Gaussian random variables. (See [3] Theorem 7.1.) First I show that under the condition  $\alpha_1(A) \leq 1$

$$E \left( \sup_{u=(u(1), \dots, u(n_1)): \sum_{i=1}^{n_1} u(i)^2 \leq 1} Y(u) \right) \leq 1. \quad (3.4)$$

Indeed, for all  $\omega \in \Omega$

$$\sup_i \sum_{u(i)^2 \leq 1} \sum_{j} a(i, j) u(i) g(j)(\omega) = \left[ \sum_i \left( \sum_j a(i, j) g(j)(\omega) \right)^2 \right]^{1/2},$$

since the above expression takes its supremum at the value

$$u(i) = \frac{\sum_j a(i, j) g(j)(\omega)}{\left[ \sum_i \left( \sum_j a(i, j) g(j)(\omega) \right)^2 \right]^{1/2}}, \quad 1 \leq i \leq n_1.$$

Hence by the Schwarz inequality and relation (3.1)

$$\begin{aligned} E \left( \sup_{u=(u(1), \dots, u(n_1)): \sum_i u(i)^2 \leq 1} Y(u) \right) &= E \left( \sup_{\sum_i u(i)^2 \leq 1} \sum_{i,j} a(i, j) u(i) g(j) \right) \\ &= E \left[ \sum_i \left( \sum_j a(i, j) g(j) \right)^2 \right]^{1/2} \leq \left[ E \sum_i \left( \sum_j a(i, j) g(j) \right)^2 \right]^{1/2} \\ &= \left( \sum_{i,j} a(i, j)^2 \right)^{1/2} = \alpha_1(A) \leq 1. \end{aligned}$$

On the other hand  $EY(u) = 0$  and  $EY(u)^2 = \alpha_2(u)^2 \leq 2^{-2N} M^{-1}$ , for all  $u \in U_N$ . Hence Ledoux's concentration inequality (see formula 7.4 in [3]) implies that

$$P \left( \sup_{u \in U_N} \left| Y(u) - E \sup_{u \in U_N} Y(u) \right| \geq x \right) \leq 2e^{-2^{-2N-1} M x^2} \quad \text{for all } x \geq 0.$$

The above inequality with partial integration yield for all  $R \geq 2$  that

$$\begin{aligned} E \sup_{u \in U_N} \left| Y(u) - E \sup_{u \in U_N} Y(u) \right|^{2R} &\leq \int_0^\infty 2e^{-2^{-2N-1} M x^2} dx^{2R} \\ &= 4R \cdot 2^{-2NR} M^{-R} \int_0^\infty x^{2R-1} e^{-x^2/2} dx = 4R \cdot 2^{-2NR} M^{-R} (2R-2)(2R-4) \cdots 2 \\ &\leq (2RM^{-1})^R 2^{-2NR} = (2RM^{-1} 2^{-2N})^R. \end{aligned}$$

Relation (3.3) follows from the above inequality with the choice  $2R = 2^{2(N+A)} M$ ,  $N \geq 0$ ,  $M \geq 1$ ,  $A \geq 1$ , and the inequality  $E \sup_{u \in U_N} Y(u) \leq 1$  which is a consequence of relation (3.4).

*Proof of the Basic estimate for  $d = 2$ .* Let us apply the Main inequality in the case  $d = 2$  with  $N = 0$  and  $A = 1$ . Since the conditions of the Basic estimate for  $d = 2$  contain the inequality  $\alpha_2(A) = \sup_{u \in B^{n_1}} \alpha_2(u) \leq M^{-1/2}$  the set  $U_0$  agrees with the unit ball  $B^{n_1}$ . Hence the Schwarz inequality and relation (3.3) with the choice  $N = 0$  and  $A = 1$  yield the estimate

$$E \left[ \sup_{u: u \in B^{n_1}} \sum_{i,j} a(i,j) u(i) g(j) \right]^{2M} \leq \left( E \left[ \sup_{u: u \in U_0} \sum_{i,j} a(i,j) u(i) g(j) \right]^{4M} \right)^{1/2} \\ \leq 4^{4M/2} = 2^{4M}.$$

The Basic estimate for  $d = 2$  (with  $C = 16$  in formula (2.6)) is proved.

#### 4. Estimates on the cardinality of $\varepsilon$ -nets with respect to nice metrics.

In the Basic estimate the moments of the supremum of a class of Gaussian random variables are estimated. In such problems it is worth introducing a natural metric on the set of parameters of the random variables we are considering, by defining the distance of two points in the parameter space as the square root of the variance of the difference of the corresponding random variables. It is also useful to find such a subset of the parameter space with relatively small cardinality which is dense with respect to this metric. Such an approach leads to the formulation of the following problem.

Given a pseudometric space  $(X, \rho)$  together with a subset  $X_0 \subset X$  we want to find for all  $\varepsilon > 0$  an  $\varepsilon$ -net of relatively small cardinality in the space  $X_0$  with respect to the metric  $\rho$ , i.e. we want to find a set  $\{x_1, \dots, x_N\} \subset X_0$  with a relatively small index  $N$  for which  $\min_{1 \leq j \leq N} \rho(x_j, x) \leq \varepsilon$  for all  $x \in X_0$ . A good  $\varepsilon$ -net can be found by solving the following problem. Let us define an appropriate probability measure  $\mu$  in the space  $(X, \rho)$  and give a good lower bound on the probability  $\mu(\{y: y \in X, \rho(y, x) \leq \varepsilon\})$  for all  $x \in X_0$  and  $\varepsilon > 0$ .

Latała presented two estimates of this kind in Lemmas 1 and 2 of his paper [2]. In Lemma 1 that case is considered when  $X$  is the  $n$ -dimensional Euclidean space  $R^n$ ,  $X_0$  is the unit ball in this space with respect to the Euclidean metric, and the pseudometric  $\rho = \rho_\alpha$  is defined by means of a pseudonorm  $\alpha$  in  $R^n$  in the usual way, i.e.  $\rho_\alpha(x, y) = \alpha(x - y)$ . Lemma 2 is a multi-linear version of this result. Here the space  $X$  is the product of some Euclidean spaces. We embed it in the tensor product of these Euclidean spaces in a natural way, and the metric  $\rho_\alpha$  in  $X$  is defined with the help of a pseudonorm in this tensor product.

Since these results play an important role in our considerations I recall them in this paper under the names Proposition 4.1 and Proposition 4.2. I shall apply a notation different from [2], and it may be hard to compare the results formulated here with their original version. Hence to make this paper self-contained I present the proof of Latała's results in an Appendix.

To formulate these results some notations have to be introduced. We denote the unit ball in the  $n$ -dimensional Euclidean space by  $B^n$ . We introduce a probability measure  $\mu_{n,t}$  depending on a parameter  $t$  in the Euclidean space  $R^n$  in the following way. Given some number  $t > 0$  let  $\mu_{n,t}$  denote the distribution of the random vector  $tG = (tg_1, \dots, tg_n)$  in  $R^n$ , where  $g_1, \dots, g_n$  are independent standard normal random variables.

**Proposition 4.1.** *Let  $\alpha_1$  and  $\alpha_2$  be two pseudonorms in  $R^n$ ,  $t > 0$  an arbitrary positive number,  $x \in B^n$  a vector in the unit ball of  $R^n$  and  $G = (g_1, \dots, g_n)$  an  $n$ -dimensional standard normal vector. Then*

$$\mu_{n,t}(\{y: y \in R^n, \alpha_1(y - x) \leq 4E\alpha_1(tG), \alpha_2(y - x) \leq 4E\alpha_2(tG)\}) \geq \frac{1}{2}e^{-1/2t^2}$$

with the above introduced probability measure  $\mu_{n,t}$ .

*Remark.* In our applications it would be enough to consider a simpler version of Proposition 4.1 where only one pseudonorm  $\alpha_1$  appears. We formulated a result with two pseudonorm, because such a result is applied in the proof of Proposition 4.2.

To formulate Proposition 4.2 some additional notations have to be introduced. Let us consider  $d$  Euclidean spaces  $R^{n_1}, \dots, R^{n_d}$  of dimension  $n_j$ ,  $1 \leq j \leq d$ , their product  $R^{n_1} \times \dots \times R^{n_d}$  and their tensor product  $R^{n_1} \otimes \dots \otimes R^{n_d}$  with some pseudonorm  $\alpha(\cdot)$  on the tensor product. We give an embedding of the product  $R^{n_1} \times \dots \times R^{n_d}$  of these Euclidean spaces into their tensor product and define with its help a pseudometric  $\rho_\alpha$  in the product space  $R^{n_1} \times \dots \times R^{n_d}$  induced by the pseudonorm  $\alpha$  on the tensor product  $R^{n_1} \otimes \dots \otimes R^{n_d}$ .

For the sake of simpler notations we shall represent the Euclidean space  $R^n$  as the space of the real valued functions  $x = (x(1), \dots, x(n))$  on the set  $\{1, \dots, n\}$ , the tensor product  $R^{n_1} \otimes \dots \otimes R^{n_d}$  of the Euclidean spaces  $R^{n_j}$ ,  $1 \leq j \leq d$ , as the space of the real valued functions  $v(i_1, \dots, i_d)$ , defined on the set of vectors  $(i_1, \dots, i_d)$ ,  $1 \leq i_j \leq n_j$ ,  $1 \leq j \leq d$ , and the product  $R^{n_1} \times \dots \times R^{n_d}$  as the space of all vectors  $x = (x_1, \dots, x_d)$ , whose elements are real valued functions  $x_j = (x_j(1), \dots, x_j(n_j))$  on the sets  $\{1, \dots, n_j\}$ ,  $1 \leq j \leq d$ .

We embed the Euclidean space  $R^{n_1} \times \dots \times R^{n_d}$  in the tensor product  $R^{n_1} \otimes \dots \otimes R^{n_d}$  with the help of the map  $A(x) = A(x_1, \dots, x_d) = x_1 \otimes \dots \otimes x_d$  from the Euclidean space  $R^{n_1} \times \dots \times R^{n_d}$  into the tensor product  $R^{n_1} \otimes \dots \otimes R^{n_d}$ , where  $x_1 \otimes \dots \otimes x_d$  is defined for a vector  $x = (x_1, \dots, x_d) \in R^{n_1} \times \dots \times R^{n_d}$  by the formula  $x_1 \otimes \dots \otimes x_d(i_1, \dots, i_d) = x_1(i_1) \dots x_d(i_d)$  for all coordinates  $(i_1, \dots, i_d)$  with  $1 \leq i_j \leq n_j$ ,  $1 \leq j \leq d$ .

Given a pseudonorm  $\alpha$  on the tensor product  $R^{n_1} \otimes \dots \otimes R^{n_d}$  define with its help the pseudometric  $\rho_\alpha$  in the space  $R^{n_1} \times \dots \times R^{n_d}$  by the formula

$$\rho_\alpha((x_1, \dots, x_d), (y_1, \dots, y_d)) = \alpha(x_1 \otimes \dots \otimes x_d - y_1 \otimes \dots \otimes y_d) \quad (4.1)$$

for all  $x = (x_1, \dots, x_d) \in R^{n_1} \times \dots \times R^{n_d}$  and  $y = (y_1, \dots, y_d) \in R^{n_1} \times \dots \times R^{n_d}$ . I shall call this  $\rho_\alpha$  the pseudometric induced by the pseudonorm  $\alpha$ .

Let us fix some  $x = (x_1, \dots, x_d) \in B^{n_1} \times \dots \times B^{n_d}$  in the product of the unit balls  $B^{n_j}$  in  $R^{n_j}$ ,  $1 \leq j \leq d$ . In Proposition 4.2 a good lower bound is given on the probability of a small neighbourhood of such a point  $x$  with respect to an appropriately defined probability measure. More explicitly, the probability  $\mu_{n_1+\dots+n_d,t}(y: y \in R^{n_1} \times \dots \times R^{n_d}, \rho_\alpha(x, y) \leq u)$  will be bounded from below for all numbers  $u > 0$  with respect to an appropriately defined Gaussian measure  $\mu_{n_1+\dots+n_d,t}$ , where  $\rho_\alpha$  is the pseudometric in  $R^{n_1} \times \dots \times R^{n_d}$  induced by a pseudonorm  $\alpha$  in  $R^{n_1} \otimes \dots \otimes R^{n_d}$  in the above way. To formulate this result some additional notations will be introduced.

Let us consider  $d$  independent standard normal vectors  $G_j = (g_j(1), \dots, g_j(n_j))$  of dimension  $n_j$ ,  $1 \leq j \leq d$ , and for all  $t > 0$  let  $\mu_{n_1+\dots+n_d,t}$  denote the distribution of the random vector  $(tG_1, \dots, tG_d)$  in the space  $R^{n_1} \times \dots \times R^{n_d}$ . Given a pseudonorm  $\alpha$  on the tensor product  $R^{n_1} \otimes \dots \otimes R^{n_d}$  of the spaces  $R^{n_j}$ ,  $1 \leq j \leq d$ , a number  $t > 0$ , some set  $I \subset \{1, \dots, d\}$ ,  $I \neq \emptyset$  and a vector  $x = (x_1, \dots, x_d) \in R^{n_1} \times \dots \times R^{n_d}$  we define the quantity

$$W_I^x(\alpha, t) = E\alpha(z_1 \otimes \dots \otimes z_d) \quad \text{where } z_j = x_j \text{ if } j \notin I \text{ and } z_j = tG_j \text{ if } j \in I \quad (4.2)$$

with the previously defined function  $z_1 \otimes \dots \otimes z_d \in R^{n_1} \otimes \dots \otimes R^{n_d}$  for  $(z_1, \dots, z_d) \in R^{n_1} \times \dots \times R^{n_d}$ . In words, we take the function  $\alpha(x_1 \otimes \dots \otimes x_d)$ , replace the coordinates  $x_j \in R^{n_j}$  by  $tG_j \in R^{n_j}$  for the indices  $j \in I$ , and take the expected value of the random variable obtained in such a way. With the help of the above quantities we can formulate Proposition 4.2.

**Proposition 4.2.** *Let us have a pseudometric  $\rho_\alpha$  in the product  $R^{n_1} \times \dots \times R^{n_d}$  of some Euclidean spaces  $R^{n_j}$ ,  $1 \leq j \leq d$ , induced by a pseudonorm  $\alpha$  in their tensor product  $R^{n_1} \otimes \dots \otimes R^{n_d}$ . Fix some vector  $x = (x_1, \dots, x_d) \in B^{n_1} \times \dots \times B^{n_d}$ , in the product of the unit balls  $B^{n_j}$  in  $R^{n_j}$ ,  $1 \leq j \leq d$ . The following inequality holds for such a vector  $x$  and an arbitrary number  $t > 0$ .*

$$\begin{aligned} \mu_{n_1+\dots+n_d,t} \left( \left\{ y: y \in R^{n_1} \times \dots \times R^{n_d}, \rho_\alpha(x, y) \leq \sum_{I: I \subset \{1, \dots, d\}, I \neq \emptyset} W_I^x(\alpha, 4t) \right\} \right) \\ \geq 2^{-d} e^{-d/2t^2} \end{aligned} \quad (4.3)$$

with the Gaussian probability measure  $\mu_{n_1+\dots+n_d,t}$  defined above.

The following corollary of Proposition 4.2 is important for us.

**Corollary of Proposition 4.2.** *Let us have a pseudometric  $\rho_\alpha$  in  $R^{n_1} \times \dots \times R^{n_d}$  induced by a pseudonorm  $\alpha$  in the tensor product  $R^{n_1} \otimes \dots \otimes R^{n_d}$  of the Euclidean spaces  $R^{n_j}$ ,  $1 \leq j \leq d$ . Let  $D \subset B^{n_1} \times \dots \times B^{n_d}$  be a subset of the product of the unit balls  $B^{n_j}$ ,  $1 \leq j \leq d$  that has the following property:  $\sum_{I \subset \{1, \dots, d\}, I \neq \emptyset} W_I^x(\alpha, 4t) \leq u$  with some fixed numbers  $0 < t \leq 1$  and  $u > 0$  for all  $x \in D$ .*

*Then there is a constant  $C > 0$  depending only on the parameter  $d$  such that the set  $D$  has a  $2u$ -net of cardinality  $e^{C/t^2}$  with respect to the pseudometric  $\rho_\alpha$ . In more detail*

this means that there is a set  $\{x^{(1)}, \dots, x^{(N)}\} \subset D$  with cardinality  $N \leq e^{C/t^2}$  such that  $\min_{1 \leq j \leq N} \rho_\alpha(x, x^{(j)}) \leq 2u$  for all  $x \in D$ .

*Proof of the Corollary.* Let us construct a sequence  $x^{(1)}, x^{(2)}, \dots, x^{(N)}, x^{(j)} \in D$ ,  $1 \leq j \leq N$ , in the following way. Let us choose first a point  $x^{(1)} \in D$  in an arbitrary way. If the points  $x^{(1)}, \dots, x^{(j)}$  are already chosen, and there are some points  $x \in D$  such that  $\rho_\alpha(x, x^{(p)}) > 2u$  for all  $1 \leq p \leq j$ , then we choose an arbitrary point  $x \in D$  with this property as  $x^{(j+1)}$ . If there is no such point, then we finish our procedure at the  $j$ -th step. Let  $N$  be the number of points  $x^{(j)}$  that we could choose in such a way. Observe that the sets  $U_j = \{y: y \in R^{n_1} \times \dots \times R^{n_d}, \rho_\alpha(y, x^{(j)}) \leq u\}$ ,  $1 \leq j \leq N$ , are disjoint, because  $\rho_\alpha(x_j, x_{j'}) > 2u$  for all  $1 \leq j, j' \leq N$ ,  $j \neq j'$ . Beside this,  $\mu_{n_1+\dots+n_d, t}(U_j) \geq 2^{-d}e^{-d/2t^2}$  by Proposition 4.2 for all  $1 \leq j \leq N$ . Hence  $N \leq 2^d e^{d/2t^2} \leq e^{C/t^2}$ . Beside this, the set  $\{x^{(1)}, \dots, x^{(N)}\}$  is a  $2u$ -net in  $D$ , because if there were a point  $x \in D$  such that  $\min_{1 \leq j \leq N} \rho_\alpha(x, x^{(j)}) > 2u$  then we would not finish our procedure at the  $N$ -th step.

*Remark.* In the proof of the above corollary we applied a rather standard method, well-known in the literature. In general applications of a result similar to Proposition 4.2 the cardinality of a good  $\varepsilon$ -net of the set  $B^{n_1} \times \dots \times B^{n_d}$  is bounded. Here a slightly more general result was proved. This corollary gave an estimate about the cardinality of a good  $\varepsilon$ -net of an arbitrary set  $D \subset B^{n_1} \times \dots \times B^{n_d}$ . For some sets  $D$  with nice properties it provides a much better bound for the cardinality of a good  $\varepsilon$ -net in  $D$  than for the cardinality of a good  $\varepsilon$ -net in  $B^{n_1} \times \dots \times B^{n_d}$ . This observation will be exploited in our further considerations.

In formula (2.11) we defined a pseudometric  $\rho_\alpha$  in the product  $R^{n_1} \times \dots \times R^{n_{d-1}}$  of the Euclidean spaces  $R^{n_j}$ ,  $1 \leq j \leq n$  and in formula (2.12) a pseudonorm  $\alpha$  in their tensor product  $R^{n_1} \otimes \dots \otimes R^{n_{d-1}}$ . A comparison of formulas (2.13) and (4.1) shows that Proposition 4.2 and its corollary can be applied (with parameter  $d-1$ ) for the metric  $\rho_\alpha$  and norm  $\alpha$  defined in (2.11) and (2.12). This fact plays an important role in the proof of the Basic estimate.



## 5. The Main inequality.

In this section I formulate a result that I call the Main inequality and show that the Basic estimate and in such a way Theorem 1 follows from it. This result is a weaker version of an inductive statement formulated in the proof of Theorem 3 in [2]. I had to formulate such a weaker statement because the corresponding result in [2] seems to be incorrect.

Let us fix the parameter  $d \geq 3$ . We shall define appropriate classes  $\mathcal{U}(r, N)$  depending on two parameters  $N$  and  $r$  which consist of finite subsets of  $R^{n_1} \times \dots \times R^{n_{d-1}}$  with some nice properties. In the Main inequality we give an estimate on the moments of the random variables  $\sup_{u \in U, u' \in U} [Y_d(u) - Y_d(u')]$  for the sets  $U \in \mathcal{U}(r, N)$ , where  $Y_d(u)$  with parameter  $u \in R^{n_1} \times \dots \times R^{n_{d-1}}$  is the Gaussian random variable defined in (2.1). To define these classes of sets  $\mathcal{U}(r, N)$  some additional quantities have to be introduced.

We shall work with the linear functional  $A(v) = A(v, d)$  defined for functions  $v \in R^{n_1} \otimes \dots \otimes R^{n_d}$  in formula (1.3) with the help of a set of numbers  $A = \{a(i_1, \dots, i_d), 1 \leq i_p \leq n_p, 1 \leq p \leq d\}$ . Let us also recall the definition of the Gaussian random variables  $Y_d(u)$  defined in (2.1) for vectors  $u = (u_1, \dots, u_{d-1}) \in R^{n_1} \times \dots \times R^{n_{d-1}}$  together with a standard Gaussian random vector  $G_d = (g_d(1), \dots, g_d(n_d))$ . We shall also work with the quantity  $\rho_\alpha(u, v)$ ,  $u \in R^{n_1} \times \dots \times R^{n_{d-1}}$  and  $v \in R^{n_1} \times \dots \times R^{n_{d-1}}$  defined in (2.11).

Beside this, to define the sets  $\mathcal{U}(r, N)$  we still have to introduce some pseudonorms  $\tilde{\alpha}_{j,k}$  in the spaces  $R^{n_j}$  for all pairs  $j, k$  such that  $1 \leq j, k \leq d-1$ ,  $j \neq k$ , with the help of the coefficients  $a(i_1, \dots, i_d)$  appearing in formula (1.3).

For this goal first we introduce the set of constants

$$b_{u_j}^{(j)}(i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_d) = \sum_{i_j: 1 \leq i_j \leq n_j} a(i_1, \dots, i_d) u_j(i_j), \quad (5.1)$$

$$1 \leq i_p \leq n_p, \quad p \in \{1, \dots, d\} \setminus \{j\},$$

for all vectors  $u_j \in R^{n_j}$  and the functional

$$B_{u_j}^{(j)}(v) = \sum_{\substack{(i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_d) \\ 1 \leq i_p \leq n_p, p \in \{1, \dots, d\} \setminus \{j\}}} b_{u_j}^{(j)}(i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_d) v(i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_d) \quad (5.2)$$

depending on this  $u_j \in R^{n_j}$  for all  $v \in R^{n_1} \otimes \dots \otimes R^{n_{j-1}} \otimes R^{n_{j+1}} \otimes \dots \otimes R^{n_d}$ . The functional  $B_{u_j}^{(j)}(v)$  defined in (5.2) is a special case of the operator  $B_K(v)$  introduced in (1.4) if we choose  $K = \{1, \dots, d\} \setminus \{j\}$  and the coefficient  $b_K(\cdot)$  are chosen as the numbers  $b_{u_j}^{(j)}(\cdot)$  introduced in (5.1). With such a choice we can introduce the quantity  $V(P, B_{u_j}^{(j)}) = V(P, b_{u_j}^{(j)}(\cdot))$  for all partitions  $P$  of the set  $\{1, \dots, d\} \setminus \{j\}$  as the quantity  $V(P, B_K) = V(P, b_K(\cdot))$  defined in (1.6) with this choice  $K = \{1, \dots, d\} \setminus \{j\}$  and  $B_K(v) = B_{u_j}^{(j)}(v)$ . Let  $P_{j,k}$  denote the partition  $P_{j,k} = \{\{k, d\}, \{l\}, 1 \leq l \leq d-1, l \neq j, k\}$  of the set  $\{1, \dots, d\} \setminus \{j\}$ , and define

$$\tilde{\alpha}_{j,k}(u_j) = V(P_{j,k}, B_{u_j}^{(j)}), \quad 1 \leq j, k \leq d-1, \quad k \neq j, \quad u_j \in R^{n_j}. \quad (5.3)$$

It is easy to check that  $\tilde{\alpha}_{j,k}(u_j)$  is a pseudonorm in  $R^{n_j}$ .

The expression  $\tilde{\alpha}_{j,k}(u_j)$  can also be written as

$$\tilde{\alpha}_{j,k}(u_j) = \sup_{\substack{v_p(\cdot), p \in \{1, \dots, d-1\} \setminus \{j, k\}, v_{k,d}(\cdot, \cdot): \\ \sum_{i_p} v_p^2(i_p) \leq 1, p \in \{1, \dots, d-1\} \setminus \{j, k\}, \sum_{i_k, i_d} v^2(i_k, i_d) \leq 1,}} \sum_{i_1, \dots, i_d} a(i_1, \dots, i_d) u_j(i_j) v(i_k, i_d) \prod_{p \in \{1, \dots, d-1\} \setminus \{j, k\}} v_p(i_p) \quad (5.4)$$

for any  $u_j = (u_j(1), \dots, u_j(n_j)) \in R^{n_j}$ .

Given an operator  $A(v)$  of order  $d$ ,  $d \geq 3$ , defined in (1.3) and a positive integer  $M$  the following classes of sets  $\mathcal{U}(r, N) = \mathcal{U}_{A, M, d}(r, N)$  consisting of at most  $r$  elements  $u \in R^{n_1} \times \dots \otimes R^{n_{d-1}}$  will be introduced.

$$\begin{aligned} \mathcal{U}(r, N) &= \mathcal{U}_{A, M, d}(r, N) \\ &= \left\{ U = \{(u^{(t)} = (u_1^{(t)}, \dots, u_{d-1}^{(t)}) \in R^{n_1} \times \dots \times R^{n_{d-1}}, 1 \leq t \leq r')\} : \right. \\ &\quad 1 \leq r' \leq r, \quad \tilde{\alpha}_{j,k}(u_j^{(t)}) \leq 2^{-N} M^{-(d-2)/2}, \text{ for all } 1 \leq t \leq r' \\ &\quad \text{and } 1 \leq j, k \leq d-1, j \neq k, \\ &\quad \rho_\alpha(u^{(t)}, u^{(t')}) \leq 2^{-2N} M^{-(d-1)/2} \text{ for all } 1 \leq t, t' \leq r', \\ &\quad u^{(t)} \in B^{n_1} \times \dots \times B^{n_{d-1}}, \text{ for all } 1 \leq t \leq r', \\ &\quad \left. u^{(t)} - u^{(t')} \in B^{n_1} \times \dots \times B^{n_{d-1}} \text{ for all } 1 \leq t, t' \leq r' \right\} \end{aligned} \quad (5.5)$$

with the above defined  $\tilde{\alpha}_{j,k}$  and the quantity  $\rho_\alpha(\cdot, \cdot)$  introduced in (2.11).

In the Main estimate we shall prove a moment estimate for the supremum of some random variables determined with the help of the sets  $\mathcal{U}(r, N)$ . It holds under the condition

$$\alpha_s = \alpha_s(A) \leq M^{-(s-1)/2}, \quad \text{for all } 1 \leq s \leq d-1, \quad (5.6)$$

where the quantities  $\alpha_s$  were defined in (1.7).

*Remark.* In Theorem 1A we imposed a similar but stronger condition in formula (1.10). It also contained the condition  $\alpha_d \leq M^{-(d-1)/2}$  for  $s = d$ . This condition is missing here. It is replaced by the inequalities imposed on  $\rho_\alpha$  in the definition of the sets  $\mathcal{U}(r, N)$ . The additional condition of Theorem 1A is needed when we want to prove the Basic estimate with the help of the Main inequality.

**The Main inequality.** *Let a multilinear form  $A$  of order  $d \geq 3$  satisfy condition (5.6). Take a standard normal random vector  $G_d = (g_d(1), \dots, g_d(n_d))$  of dimension  $n_d$ , and introduce with its help the random variables  $Y_d(u)$  defined in (2.1) for all vectors  $u =$*

$(u_1, \dots, u_{d-1}), u_p = (u_p(1), \dots, u_p(n_p)) \in R^{n_p}, 1 \leq p \leq d-1$ . There is a threshold index  $A_0 \geq 0$  and a constant  $C > 0$  such that for integers  $r \geq 1$  and  $N \geq 0$  the inequality

$$E \left[ \sup_{\substack{u^{(t)} = (u_1^{(t)}, \dots, u_{d-1}^{(t)}) \in U, \\ u^{(t')} = (u_1^{(t')}, \dots, u_{d-1}^{(t')}) \in U}} (Y_d(u^{(t)}) - Y_d(u^{(t')})) \right]^{2^{2(N+A)}M} \leq (CM^{-(d-2)/2} 2^{(A-N)})^{2^{2(N+A)}M} \quad (5.7)$$

holds for all  $U \in \mathcal{U}(r, N)$  and integers  $A \geq A_0$ . The numbers  $A_0 = A_0(d)$  and  $C = C(d)$  are sufficiently large constants which depend only on  $d$  and do not depend on the parameters  $r$  and  $N$ .

Now I give the proof of the Basic estimate with the help of the Main inequality.

*The proof of the Basic estimate.* First we show that under the conditions of the Basic estimate  $U \in \mathcal{U}(r, 0)$  for any set  $U = \{(u^{(t)}, 1 \leq t \leq r\}$  consisting of  $r$  vectors  $u^{(t)} = (u_1^{(t)}, \dots, u_{d-1}^{(t)})$ ,  $1 \leq t \leq r$  such that  $2u_j^{(t)} \in B^{n_j}$  for all  $1 \leq j \leq d-1$ ,  $1 \leq t \leq r$ . To show this observe that

$$\rho_\alpha(u^{(t)}, u^{(t')}) \leq \rho_\alpha(u^{(t)}, 0) + \rho_\alpha(u^{(t')}, 0)$$

for all  $1 \leq t, t' \leq r$ , where 0 denotes the vector with all coordinates 0, and

$$\begin{aligned} \rho_\alpha(u^{(t)}, 0) &\leq 2^{-(d-1)} \sup_{\substack{u = (u_1, \dots, u_{d-1}): \\ u_j \in B^{n_j}, 1 \leq j \leq d-1}} \rho_\alpha(u, 0) \\ &= 2^{-(d-1)} \sup_{\substack{u = (u_1, \dots, u_{d-1}): \\ u_j \in B^{n_j}, 1 \leq j \leq d-1}} \left[ \sum_{i_d} \left( \sum_{i_1, \dots, i_{d-1}} a(i_1, \dots, i_d) \prod_{j=1}^{d-1} u_j(i_j) \right)^2 \right]^{1/2} \\ &= 2^{-(d-1)} \sup_{\substack{u = (u_1, \dots, u_d): \\ u_j \in B^{n_j}, 1 \leq j \leq d}} \sum_{i_1, \dots, i_d} a(i_1, \dots, i_d) \prod_{j=1}^d u_j(i_j) = 2^{-(d-1)} \alpha_d \\ &\leq 2^{-(d-1)} M^{-(d-1)/2} \end{aligned}$$

for all  $1 \leq t \leq r$ , and a similar estimate holds for  $\rho_\alpha(u^{(t')}, 0)$ . (This is the point where we exploited that the estimate  $\alpha_s \leq M^{-(s-1)/2}$  also holds for  $s = d$ .) Beside this  $\alpha_{j,k}(u_j^{(t)}) \leq \frac{1}{2} \alpha_{d-1} \leq \frac{1}{2} M^{-(d-2)/2}$ , and clearly  $u^{(t)} \in B^{n_1} \times \dots \times B^{n_d}$ , and  $u^{(t)} - u^{(t')} \in B^{n_1} \times \dots \times B^{n_d}$  for all  $1 \leq t, t' \leq r$ . The above relations imply that  $U \in \mathcal{U}(r, 0)$ .

It can be proved with the help of the above fact and the Main inequality with the choice  $N = 0$  that

$$E \left[ \sup_{u \in \frac{1}{2} B^{n_1} \times \dots \times \frac{1}{2} B^{n_{d-1}}} Y_d(u) \right]^{2^{2A_0}M} \leq \left( C 2^{2A_0} M^{(d-2)/2} \right)^{2^{2A_0}M} \quad (5.8)$$

with the same number  $A_0$  which appears in the Main inequality as the threshold index.

To prove this statement let us list the set of vectors  $u \in \frac{1}{2}B^{n_1} \times \dots \times \frac{1}{2}B^{n_{d-1}}$  such that all their coordinates are rational numbers in a sequence  $u^{(t)}$ ,  $t = 1, 2, \dots$ . Let  $u^{(1)} = (0, \dots, 0)$  in this sequence. Let  $U_r = \{u^{(t)}, 1 \leq t \leq r\}$  be the set consisting of the first  $r$  terms of this sequence. Observe that

$$\sup_{u \in B^{n_1} \times \dots \times B^{n_{d-1}}} Y_d(u) = \lim_{r \rightarrow \infty} \sup_{u^{(t)} \in U_r} Y_d(u^{(t)})$$

Let us apply a weakened form of the Main inequality with  $N = 0$  and  $A = A_0$  (we may assume that  $A_0 \geq 1$ ) for all above defined sets  $U_r$ ,  $r = 1, 2, \dots$ , where instead of taking the supremum of all differences  $Y_d(u^{(t)}) - Y_d(u^{(t')})$ ,  $1 \leq t, t' \leq r$  we take this supremum only for pairs  $(t, t')$  with  $t' = 1$ . In this case  $Y_d(u^{(t')}) = 0$  with probability 1. The series of inequalities obtained in such a way, (where the upper bound does not depend on  $r$ ) together with the previous identity and the Beppo-Levy theorem imply relation (5.8). This inequality together with the Hölder inequality for  $p = 2^{2A_0-1}$  yield that

$$E \left[ \sup_{u \in \frac{1}{2}B^{n_1} \times \dots \times \frac{1}{2}B^{n_{d-1}}} Y_d(u) \right]^{2M} \leq \left( C 2^{A_0} M^{(d-2)/2} \right)^{2M} \leq \bar{C}^M M^{-(d-2)M} \quad (5.9)$$

with a universal constant  $\bar{C}$ . Relation (2.6) follows from this inequality. To see this it is enough to observe that if the condition  $u \in \frac{1}{2}B^{n_1} \times \dots \times \frac{1}{2}B^{n_{d-1}}$  is replaced by the condition  $u \in B^{n_1} \times \dots \times B^{n_{d-1}}$ , then the inequality remains valid if the right-hand side is multiplied by  $2^{(d-1)M}$ , i.e. the constant  $\bar{C}$  is multiplied by  $2^{(d-1)}$  in (5.9).

*Remark.* Actually we needed the Main inequality only for  $N = 0$  (and arbitrary  $r$ ). But we shall prove it by a backward induction procedure. It is not difficult to see that the Main inequality holds if  $N \geq N_0$  with a very large  $N_0$  whose value may depend on  $r$ . If this is shown, then we may apply backward induction to prove the Main inequality. It may seem a technical point that the hardest estimate of this paper is proved by a backward and not by a forward induction. But I think that the situation is much more complex.

I met a similar situation in a study leading to paper [5]. Here also backward induction had to be applied to solve the hardest part of the problem, and this had a non-technical reason. The supremum of such random variables had to be bounded whose behaviour was very ‘non-Gaussian’. The main contribution to the supremum I was interested in came from the influence of some irregular events. These irregular events had very small probability, but they played a dominant role because of their large number. Their effect could be controlled by means of a backward and not by a forward induction procedure. I believe that behind the proof of the Main inequality in this paper a similar phenomenon is hiding. But to understand the situation better some additional work has to be done.

## 6. Some results about the existence of good partitions.

The proof of the Main inequality is based on the existence of some good partitions of the class of sets  $\mathcal{U}(r, N)$  defined in (5.5). These results are formulated in this section in Lemmas 6.1 and 6.2. Lemma 6.1 can be considered as a version of Lemma 8 in [2], and Lemma 6.2 is an improvement of this result. It states that there exists a partition of the sets  $U \in \mathcal{U}(r, N)$  which satisfies Lemma 6.1, and it also has some extra properties useful in our investigation. Its cardinality can be bounded similarly to Lemma 6.1. Such a result was needed to get a proof without the application of Theorem 3 of [2] whose validity is questionable. Lemmas 6.1 and 6.2 are proved by means of Proposition 4.2 and its corollary. But to prove them we also need some additional inequalities. They are given in Lemma 6.3 which can be considered as a version of Lemmas 5 and 6 in [2]. Lemma 6.3 is formulated and proved in this section.

Before the formulation of these results some additional notations have to be introduced. We define with the help of a vector  $u \in R^{n_1} \times \dots \times R^{n_{d-1}}$  and a set  $I \subset \{1, \dots, d-1\}$  an operator which is a special case of the class of operators defined in formula (1.4). We also introduce some quantities corresponding to this operator which are the analogs of the quantities  $\alpha_s, \tilde{\alpha}_{j,k}, \rho_\alpha$  defined earlier with the help of the operator  $A(v)$  given in (1.3).

Fix a set  $I = \{j_1, \dots, j_s\} \subset \{1, \dots, d-1\}$  with  $1 \leq s \leq d-2$  elements and a vector  $u = (u_1, \dots, u_{d-1}) \in R^{n_1} \times \dots \times R^{n_{d-1}}$ . Let us define with their help the numbers

$$b_u^I(i_j, j \in \{1, \dots, d\} \setminus I) = M^{|I|/2} \sum_{(i_j, j \in I)} a(i_1, \dots, i_d) \prod_{j \in I} u_j(i_j) \quad (6.1)$$

depending on the vectors  $(i_j, j \in \{1, \dots, d\} \setminus I)$  and the linear functional

$$B_u^I(v) = \sum_{(i_j, j \in \{1, \dots, d\} \setminus I)} b_u^I(i_j, j \in \{1, \dots, d\} \setminus I) v(i_j, j \in \{1, \dots, d\} \setminus I) \quad (6.2)$$

acting on the space of functions  $v = v(i_{j_1}, \dots, i_{j_p}) \in R^{n_{j_1}} \otimes \dots \otimes R^{n_{j_p}}$ , with the set of indices  $\{j_1, \dots, j_p\} = \{1, \dots, d\} \setminus I$ .

This operator  $B_u^I(v)$  is a special case of the operators  $B_K(v)$  defined in formula (1.4) when  $K = \{1, \dots, d\} \setminus I$ , and coefficients  $b_K(\cdot)$  are the numbers  $b_u^I(\cdot)$  defined in (6.1). (In the definition of the coefficients  $b_u^I(i_j, j \in \{1, \dots, d\} \setminus I)$  in (6.1) a multiplying factor  $M^{|I|/2}$  was inserted. I applied such a norming factor, because it simplifies the subsequent calculations.)

We can define the quantities  $V(P, B_u^I)$  for all partitions  $P$  of the set  $K = \{1, \dots, d\} \setminus I$  by formula (1.6) with the choice  $B_K(v) = B_u^I(v)$ . Let us also introduce, similarly to  $\alpha_s$  defined in (1.7) the quantity

$$\alpha_{u,s}(I) = \sup_{P: |P|=s} V(P, B_u^I), \quad (6.3)$$

where all partitions  $P$  of the set  $\{1, \dots, d\} \setminus I$  with cardinality  $s$  are taking part in the supremum. We also introduce the numbers

$$\tilde{\alpha}_k^I(u) = V(P_{I,k}, B_u^I) \quad \text{for all } k \in \{1, \dots, d-1\} \setminus I \quad (6.4)$$

with the help of formula (1.6), where the operator  $B_u^I(v)$  defined in (6.2) plays the role of  $B_K(v)$ , and the partition  $P_{I,k}$  of the set  $\{1, \dots, d\} \setminus I$  is defined as  $P_{I,k} = \{\{k, d\}, \{l\}, l \in \{1, \dots, d-1\} \setminus (I \cup \{k\})\}$ .

We shall also work with a quantity  $\rho_{\alpha_u^I}(v, \bar{v})$  defined for all pairs  $(v, \bar{v})$ ,  $v \in R^{n_1} \times \dots \times R^{n_{d-1}}$  and  $\bar{v} \in R^{n_1} \times \dots \times R^{n_{d-1}}$  with the help of a vector  $u \in R^{n_1} \times \dots \times R^{n_{d-1}}$  and set  $I \subset \{1, \dots, d-1\}$ ,  $1 \leq |I| \leq d-2$ , similarly to the term  $\rho_\alpha$  introduced in (2.11). First we define a version of it.

$$\begin{aligned} \bar{\rho}_{\alpha_u^I}(v, \bar{v}) = & \left( \sum_{i_d} \left[ \sum_{(i_j, j \in \{1, \dots, d-1\} \setminus I)} b_u^I(i_j, j \in \{1, \dots, d\} \setminus I) \right. \right. \\ & \left. \left. \left( \prod_{j \in \{1, \dots, d-1\} \setminus I} v(i_j) - \prod_{j \in \{1, \dots, d-1\} \setminus I} \bar{v}(i_j) \right) \right]^2 \right)^{1/2} \end{aligned} \quad (6.5)$$

for pairs of vectors  $v = (v_{j_1}, \dots, v_{j_p}) \in R^{n_{j_1}} \times \dots \times R^{n_{j_p}}$  and  $\bar{v} = (\bar{v}_{j_1}, \dots, \bar{v}_{j_p}) \in R^{n_{j_1}} \times \dots \times R^{n_{j_p}}$ , where  $\{j_1, \dots, j_p\} = \{1, \dots, d-1\} \setminus I$ . Observe that  $\bar{\rho}_{\alpha_u^I}$  is the pseudometric induced by the pseudonorm

$$\begin{aligned} \alpha_u^I(v) &= \alpha_u^I(v(i_j, j \in \{1, \dots, d-1\} \setminus I)) \\ &= \left( \sum_{i_d} \left[ \sum_{(i_j, j \in \{1, \dots, d-1\} \setminus I)} b_u^I(i_j, j \in \{1, \dots, d\} \setminus I) \right. \right. \\ & \quad \left. \left. v(i_j, j \in \{1, \dots, d-1\} \setminus I) \right]^2 \right)^{1/2} \end{aligned} \quad (6.6)$$

on the tensor product  $R^{n_{j_1}} \otimes \dots \otimes R^{n_{j_p}}$ , where  $\{j_1, \dots, j_p\} = \{1, \dots, d-1\} \setminus I$ .

We can define the metric  $\rho_{\alpha_u^I}$  in the space  $R^{n_1} \times \dots \times R^{n_{d-1}}$  with the help of the metric  $\bar{\rho}_{\alpha_u^I}$  defined in (6.5). To do this we introduce the following notation. Given a vector  $v = (v_1, \dots, v_{d-1}) \in R^{n_1} \times \dots \times R^{n_{d-1}}$  and a set  $I \subset \{1, \dots, d-1\}$  let  $v_{I^c}$  denote the vector we obtain by omitting the coordinates of the vector  $v$  belonging to the set  $I$ , i.e. let  $v_{I^c} \in R^{n_{j_1}} \times \dots \times R^{n_{j_p}}$ , and  $v_{I^c} = (v_j, j \in \{1, \dots, d-1\} \setminus I)$ . Given two vectors  $v = (v_1, \dots, v_{d-1}) \in R^{n_1} \times \dots \times R^{n_{d-1}}$  and  $\bar{v} = (\bar{v}_1, \dots, \bar{v}_{d-1}) \in R^{n_1} \times \dots \times R^{n_{d-1}}$  put

$$\rho_{\alpha_u^I}(v, \bar{v}) = \bar{\rho}_{\alpha_u^I}(v_{I^c}, \bar{v}_{I^c}). \quad (6.7)$$

Now I formulate Lemma 6.1 and its strengthened version Lemma 6.2.

**Lemma 6.1.** *If an operator  $A$  of order  $d \geq 3$  satisfies relation (5.6), then each set  $U \in \mathcal{U}(r, N) = \mathcal{U}_{A,M,d}(r, N)$  has a partition  $u^{(1)} + U_1, u^{(2)} + U_2, \dots, u^{(L)} + U_L$  with*

$L \leq 2^{C(d)M^{2^{2N}}}$  elements such that  $U_l \in \mathcal{U}(r, N+2)$  and  $u^{(l)} \in U$  for all  $1 \leq l \leq L$ . The number  $C(d)$  depends only on the order  $d$  of the operator  $A$ .

**Lemma 6.2.** *Under the conditions of Lemma 6.1 each set  $U \in \mathcal{U}(r, N)$  has a partition  $u^{(1)} + U_1, u^{(2)} + U_2, \dots, u^{(L)} + U_L$  with  $L \leq 2^{C(d)M^{2^{2N}}}$  elements such that  $u^{(l)} \in U$ ,  $U_l \in \mathcal{U}(r, N+2)$ ,  $1 \leq l \leq L$ , and it also satisfies following additional property. The inequality  $\rho_{\alpha_{u^{(l)}}^I}(u, \bar{u}) \leq 2^{-2N}M^{-(d-|I|-1)/2}$  holds for all sets  $I \subset \{1, \dots, d-1\}$ ,  $1 \leq |I| < d-2$ , and pairs of elements  $u \in U_l$  and  $\bar{u} \in U_l$ ,  $1 \leq l \leq L$ . The vector  $u^{(l)}$  in this inequality is the same vector which appears in the definition of the element  $u^{(l)} + U_l$  of the partition of  $U$ . The quantity  $\rho_{\alpha_u^I}(\cdot, \cdot)$  was defined in (6.5) and (6.7).*

Lemmas 6.1 and 6.2 will be proved with the help of the following Lemma 6.3.

**Lemma 6.3.** *Let a functional  $A(v)$  of order  $d \geq 3$  defined in (1.3) satisfy condition (5.6). Then for any  $u = (u_1, \dots, u_{d-1}) \in B^{n_1} \times \dots \times B^{n_{d-1}}$  the quantities  $W_I^u(\alpha, t)$ ,  $I \in \{1, \dots, d-1\}$ ,  $I \neq \emptyset$ , defined in (4.2) with the pseudonorm  $\alpha$  introduced in (2.12) satisfy the following inequalities.*

$$W_I^u(\alpha, t) \leq \frac{t^{|I|}}{M^{(d-|I|-1)/2}} \quad \text{if } 2 \leq |I| \leq d-1. \quad (6.8)$$

For a set  $I = \{k\}$  containing one element

$$W_{\{k\}}^u(\alpha, t) \leq t \min_{1 \leq j \leq d-1, j \neq k} \tilde{\alpha}_{j,k}(u_j), \quad (6.9)$$

where  $\tilde{\alpha}_{j,k}(u_j)$  was defined in (5.3). Beside this,

$$E\tilde{\alpha}_{j,k}(G_j) \leq \frac{C(d)}{M^{(d-3)/2}} \quad \text{for all } 1 \leq j, k \leq d-1, j \neq k, \quad (6.10)$$

where  $C(d)$  depends only on  $d$ , and  $G_j$  is a standard normal vector of dimension  $n_j$ .

The proof of Lemma 6.3. For any set  $I \subset \{1, \dots, d-1\}$ ,  $I \neq \emptyset$  and  $u \in B^{n_1} \times \dots \times B^{n_{d-1}}$

$$\begin{aligned} W_I^u(\alpha, 1) &= E \left( \left[ \sum_{i_d} \left( \sum_{i_1, \dots, i_{d-1}} a(i_1, \dots, i_d) \prod_{j \in \{1, \dots, d-1\} \setminus I} u_j(i_j) \prod_{j \in I} g_j(i_j) \right)^2 \right] \right)^{1/2} \\ &\leq \left( E \left[ \sum_{i_d} \left( \sum_{i_1, \dots, i_{d-1}} a(i_1, \dots, i_d) \prod_{j \in \{1, \dots, d-1\} \setminus I} u_j(i_j) \prod_{j \in I} g_j(i_j) \right)^2 \right] \right)^{1/2} \\ &= \left[ \sum_{(i_p, p \in I \cup \{d\})} \left( \sum_{(i_j, j \in \{1, \dots, d-1\} \setminus I)} a(i_1, \dots, i_d) \prod_{j \in \{1, \dots, d-1\} \setminus I} u_j(i_j) \right)^2 \right]^{1/2} \\ &= \sup_{\substack{v(i_p, p \in I \cup \{d\}): \\ \sum v^2(i_p, p \in I \cup \{d\}) \leq 1}} \sum_{i_1, \dots, i_d} a(i_1, \dots, i_d) v(i_p, p \in I \cup \{d\}) \prod_{j \in \{1, \dots, d-1\} \setminus I} u_j(i_j) \\ &\leq V(P_I, A), \end{aligned} \quad (6.11)$$

where  $P_I$  is the partition  $P_I = \{I \cup \{d\}, \{j\}, 1 \leq j \leq d-1, j \notin I\}$  of the set  $\{1, \dots, d\}$ , and  $V(P, A)$  is defined in (1.6).

Since the partition  $P_I$  has  $d - |I|$  elements this inequality together with relation (5.6) imply that for  $|I| \geq 2$

$$W_I^u(\alpha, t) = t^{|I|} W_I^u(\alpha, 1) \leq t^{|I|} \alpha_{P_I}(A) \leq \frac{t^{|I|}}{M^{(d-|I|-1)/2}},$$

i.e. (6.8) holds. In the case  $I = \{k\}$  we get from the last but one bound in (6.11), the representation of  $\tilde{\alpha}_{j,k}(u_j)$  in formula (5.4) and the choice of an arbitrary point  $j \in \{1, \dots, d-1\} \setminus \{k\}$  that  $W_I^u(\alpha, 1) \leq \tilde{\alpha}_{j,k}(u_j)$ , and this relation implies formula (6.9).

Inequality (6.10) can be deduced from inequality (2.6) in the Basic estimate with parameter  $d-1$  if we write up the expression  $\tilde{\alpha}_{j,k}(G_j)$  in the form (5.4), (by replacing the vector  $u_j$  by  $G_j$  in it), consider it as an expression of the form (2.1) with  $d-1$  variables by taking the pair  $(k, d)$  as one variable. Let us observe that relation (5.6) implies relation (1.10) with parameter  $d-1$  in this case, hence we may apply the Basic estimate. Let us apply a reindexation of the arguments by which the  $j$ -th variable turns to the  $d-1$ -th coordinate. The Basic estimate remains valid after such a reindexation. Since in the proof of Lemma 6.3 for parameter  $d$  we may assume that the Basic estimate holds for  $d-1$  we get inequality (6.10) from the Basic estimate and the estimate  $EZ_d \leq (EZ_d^{2M})^{1/2M}$  which is a consequence of Hölder's inequality.

## 7. The proof of Lemmas 6.1 and 6.2 about the existence of good partitions.

In this section Lemmas 6.1 and 6.2 will be proved with the help of Proposition 4.2, its corollary and Lemma 6.3.

*The proof of Lemma 6.1.* If relation (5.6) holds, then relation (6.10) in Lemma 6.3 implies the inequality  $E\tilde{\alpha}_{j,k}(G_j) \leq CM^{-(d-3)/2}$  for all  $1 \leq j, k \leq d-1, j \neq k$ . Hence Proposition 4.1 yields the estimate  $\mu_{n_j, t}(y: y \in R^{n_j}, \tilde{\alpha}_{j,k}(x-y) \leq CtM^{-(d-3)/2}) \geq e^{-C'/t^2 M^{(d-3)}}$  for all numbers  $t > 0$ , pairs  $(j, k)$ ,  $1 \leq j, k \leq d-1, j \neq k$ , and  $x \in B^{n_j}$ , where  $\mu_{n_j, t}$  denotes the distribution of  $tG_j$  if  $G_j$  is a standard normal vector of dimension  $n_j$ . This estimate, or in a simpler way corollary of Proposition 4.2 yields the following result for the metric  $\rho_\alpha(x, y) = \tilde{\alpha}_{j,k}(x-y)$  in the space  $R^{n_j}$  with the choice  $D = B^{n_j}$ ,  $t = C2^{-N}M^{-1/2}$  and  $u = 2^{-(N+3)}M^{-(d-2)/2}$

For all pairs  $(j, k)$ ,  $1 \leq j, k \leq d-1, j \neq k$ , the unit ball  $B^{n_j} \subset R^{n_j}$  has a partition  $\hat{U}_1^{(j,k)}, \dots, \hat{U}_{L(j,k)}^{(j,k)}$  with  $L(j, k) \leq e^{C/t^2} \leq 2^{C2^{2N}M}$  elements such that  $\tilde{\alpha}_{j,k}(y-x) \leq 2^{-(N+2)}M^{-(d-2)/2}$  if  $x \in \hat{U}_l^{(j,k)}$  and  $y \in \hat{U}_l^{(j,k)}$  with the same index  $l$ . Hence any set  $U \subset B^{n_1} \times \dots \times B^{n_{d-1}}$ , in particular any set  $U \in \mathcal{U}(r, N)$  has a partition  $U_1^{(j,k)}, \dots, U_{L(j,k)}^{(j,k)}$  with  $L(j, k) \leq 2^{C2^{2N}M}$  elements such that  $\tilde{\alpha}_{j,k}(x_j - y_j) \leq 2^{-(N+2)}M^{-(d-2)/2}$  if  $x = (x_1, \dots, x_{d-1}) \in U_l^{(j,k)}$  and  $y = (y_1, \dots, y_{d-1}) \in U_l^{(j,k)}$  with the same index  $1 \leq l \leq L(j, k)$ . Indeed, the sets  $U_l^{(j,k)} = \{y = (y_1, \dots, y_{d-1}): y \in U, y_j \in \hat{U}_l^{(j,k)}\}$ ,  $1 \leq l \leq L(j, k)$ , provide such a partition of  $U$ .



I claim that the existence of such partitions for all pairs  $(j, k)$ ,  $1 \leq j, k \leq d-1$ ,  $j \neq k$ , implies that each set  $U \in \mathcal{U}(r, N)$  has a partition of the form  $u^{(1)} + \bar{U}_1, u^{(2)} + \bar{U}_2, \dots, u^{(L)} + \bar{U}_L$  with  $L \leq 2^{C(d)M2^{2N}}$  elements such that  $u^{(l)} \in U$ , and  $\tilde{\alpha}_{j,k}(u_j) \leq 2^{-(N+2)}M^{-(d-2)/2}$  if  $u = (u_1, \dots, u_{d-1}) \in \bar{U}_l$  with some  $1 \leq l \leq L$  for all  $1 \leq j, k \leq d-1$ ,  $j \neq k$ .

To show this let us consider for all pairs  $(j, k)$ ,  $1 \leq j, k \leq d-1$ ,  $j \neq k$ , a partition  $U_1^{(j,k)}, \dots, U_{L(j,k)}^{(j,k)}$  of the set  $U$  with  $L(j,k) \leq 2^{C2^{2N}M}$  elements such that  $\tilde{\alpha}_{j,k}(x_j - y_j) \leq 2t$  if  $x = (x_1, \dots, x_{d-1}) \in U_l^{(j,k)}$  and  $y = (y_1, \dots, y_{d-1}) \in U_l^{(j,k)}$  with the same index  $l = l(j, k)$ . Take all intersections of the form  $\bigcap_{(j,k): 1 \leq j, k \leq d-1, j \neq k} U_{l(j,k)}^{(j,k)}$ , i.e. take

all possible intersections which contain exactly one element from each of the above partitions indexed by the pairs  $(j, k)$ . By reindexing the sets obtained in such a way we get a partition  $\tilde{U}_1, \dots, \tilde{U}_L$  of the set  $U$  with  $L \leq 2^{C2^{2N}M}$  elements such that for all pairs  $u = (u_1, \dots, u_{d-1}) \in \tilde{U}_l$  and  $\bar{u} = (\bar{u}_1, \dots, \bar{u}_{d-1}) \in \tilde{U}_l$  with the same index  $l$  and  $1 \leq j, k \leq d-1$ ,  $j \neq k$ ,  $\tilde{\alpha}_{j,k}(u_j - \bar{u}_j) \leq 2^{-(N+2)}M^{-(d-2)/2}$ . Then choosing an arbitrary element  $u^{(l)} \in \tilde{U}_l$  and writing  $\tilde{U}_l = u^{(l)} + \bar{U}_l$  with  $\bar{U}_l = \{u - u^{(l)}: u \in \tilde{U}_l\}$  we get a partition with the desired property.

It can be shown with the help of the corollary of Proposition 4.2 with the choice that each set  $\bar{U}_l$ , taking part in the above constructed partition  $u^{(l)} + \bar{U}_l$ ,  $1 \leq l \leq L$ , of the set  $U$  has a partition  $U_{l,1}, \dots, U_{l,L_l}$  with  $L_l \leq 2^{C2^{2N}M}$  elements such that  $\rho_\alpha(u, \bar{u}) \leq 2^{-(N+2)}M^{-(d-1)/2}$  if  $u \in U_{l,p}$  and  $\bar{u} \in U_{l,p}$  with the same parameters  $l$  and  $p$ . Indeed, let us choose  $t = c2^{-N}M^{-1/2}$  with a sufficiently small constant  $1 \geq c > 0$ . Observe that with the choice of such a number  $t$  and a vector  $u \in \bar{U}_l$  with some index  $1 \leq l \leq L$  we can write by (6.8)

$$W_I^u(\alpha, t) \leq \frac{t^{|I|}}{M^{(d-|I|-1)/2}} \leq c^2 2^{-2N} M^{-(d-1)/2}$$

for all sets  $I \subset \{1, \dots, d-1\}$  such that  $|I| \geq 2$ . For a set  $I = \{k\}$ ,  $1 \leq k \leq d-1$ , containing one element we have

$$W_{\{k\}}^u(\alpha, t) \leq t \min_{1 \leq j \leq d-1, j \neq k} \tilde{\alpha}_{j,k}(u_j) \leq c2^{-2N} M^{-(d-1)/2}$$

by relations (6.9) and  $\tilde{\alpha}_{j,k}(u_j) \leq 2^{-(N+2)}M^{-(d-2)/2}$  if  $u = (u_1, \dots, u_{d-1}) \in \bar{U}_l$ . Hence

$$\sum_{I: I \subset \{1, \dots, d-1\}, I \neq \emptyset} W_I^u(\alpha, 4t) \leq 2^{-2(N+3)} M^{-(d-1)/2}$$

for a vector  $u \in \bar{U}_l$  if the parameter  $c > 0$  is chosen sufficiently small. Then an application of the corollary of Proposition 4.2 for one of the sets  $\bar{U}_l$ ,  $1 \leq l \leq L$  with the metric  $\rho_\alpha$  and the choice  $t = c2^{-N}M^{-1/2}$  and  $u = 2^{-(N+3)}M^{-(d-1)/2}$  shows that there exists a partition  $U_{l,p}$ ,  $1 \leq p \leq L_l$ , of  $\bar{U}_l$  of cardinality  $L_l \leq 2^{C_1/t^2} \leq 2^{CM2^{2N}}$  with the desired property.

Put  $u^{(l,p)} = u^{(l)}$  for all  $1 \leq l \leq L$  and  $1 \leq p \leq L_l$ , and consider all sets  $u^{(l,p)} + U_{l,p}$ ,  $1 \leq l \leq L$ ,  $1 \leq p \leq L_l$ . I claim that a reindexation of these sets provides a partition of the set  $U \in \mathcal{U}(r, N)$  that satisfies Lemma 6.1. Indeed, these sets provide a partition of the set  $U$  with  $L \leq 2^{CM2^{2N}}$  elements. Beside this,  $u^{(l,p)} \in U$  for all indices  $l$  and  $p$ . We still have to check that  $U_{l,p} \in \mathcal{U}(r, N+2)$  for all pairs of indices  $l$  and  $p$ . The elements of the sets  $U_{l,p}$  satisfy the desired inequalities for  $\tilde{\alpha}_{j,k}$  and  $\rho_\alpha$ , and the sets  $U_{l,p}$  have at most  $r$  elements. To check that the sets  $U_{l,p}$  satisfy the remaining properties of the elements of the class  $\mathcal{U}(r, N+2)$  observe that for a point  $u \in U_{l,p}$   $u = \tilde{u} - u^{(l)}$  with  $\tilde{u} \in \tilde{U}_{l,p} \subset U$  and  $u^{(l)} \in U$ , hence  $u \in B^{n_1} \times \dots \times B^{n_{d-1}}$ . The analogous statement also holds for a difference  $u - u'$  with  $u \in U_{l,p}$  and  $u' \in U_{l,p}$ , since such a difference can be written as the difference of two vectors from the set  $\tilde{U}_l \subset U$ .

*The proof of Lemma 6.2.* The main step of the proof is the verification of the following statement formulated in relation (7.1).

Take a partition  $u^{(l)} + U_l$ ,  $1 \leq l \leq L$ , of a set  $U \in \mathcal{U}(r, N)$  that satisfies Lemma 6.1, and fix one of the vectors  $u^{(l)}$  in this partition together with a set  $I \subset \{1, \dots, d-1\}$ ,  $1 \leq |I| \leq d-2$ . There is a partition  $V_1 = V_1(l, I), \dots, V_L = V_L(l, I)$  with  $L(l, I) \leq 2^{C2^{2N}M}$  elements of the product of unit balls  $B^{n_{j_1}} \times \dots \times B^{n_{j_r}}$  with indices  $\{j_1, \dots, j_r\} = \{1, \dots, d-1\} \setminus I$  such that

$$\bar{\rho}_{\alpha_{u^{(l)}}}^{I_u}(v, \bar{v}) \leq 2^{-2N} M^{-(d-|I|-1)/2} \quad \text{if } v \in V_p(l, I) \text{ and } \bar{v} \in V_p(l, I) \text{ with an index } p, \quad (7.1)$$

i.e. this inequality holds if  $v$  and  $\bar{v}$  are contained in the same element of the partition  $V_p(l, I)$ ,  $1 \leq p \leq L(l, I)$ , of the set  $B^{n_{j_1}} \times \dots \times B^{n_{j_r}}$ . The metric  $\bar{\rho}_{\alpha_u^I}(v, \bar{v})$  (with a general vector  $u \in R^{n_1} \times \dots \times R^{n_{d-1}}$ ) was defined in formula (6.5).

First the following inequalities will be verified. For all sets  $I$ ,  $I \subset \{1, \dots, d-1\}$ ,  $1 \leq |I| \leq d-2$

$$\alpha_{u^{(l)}, s}(I) \leq M^{-(s-1)/2} \quad \text{for all } 1 \leq s \leq d - |I| - 1 \quad (7.2)$$

and

$$\tilde{\alpha}_k^I(u^{(l)}) \leq 2^{-N} M^{-(d-|I|-2)/2} \quad \text{for all } k \in \{1, \dots, d-1\} \setminus I, \quad (7.3)$$

where  $\alpha_{u, s}(I)$  was defined in (6.3) and  $\tilde{\alpha}_k^I(u)$  in (6.4) (for a general vector  $u$ ).

To check (7.2) let us compare a partition  $P$  of  $\{1, \dots, d\} \setminus I$  of cardinality  $|P| = s$ ,  $1 \leq s \leq d - |I| - 1$ , with the partition  $\bar{P}$  of the set  $\{1, \dots, d\}$  we get by attaching all one point sets of  $I$  to the elements of the partition  $P$ . Then  $|\bar{P}| = s + |I|$ , hence  $V(\bar{P}, A) \leq \alpha_{s+|I|}(A) \leq M^{-(s+|I|-1)/2}$  by relation (5.6) and  $V(P, B_{u^{(l)}}^I) \leq M^{|I|/2} V(\bar{P}, A) \leq M^{-(s-1)/2}$ . Since this relation holds for all partitions  $P$  such that  $|P| = s$  this implies (7.2).

Beside this the relation  $u^{(l)} \in U$  with an  $U \in \mathcal{U}(r, N)$  implies that  $\tilde{\alpha}_{j,k}(u_j^{(l)}) \leq 2^{-N} M^{-(d-2)/2}$ , and  $\tilde{\alpha}_k^I(u^{(l)}) \leq M^{|I|/2} \tilde{\alpha}_{j,k}(u_j^{(l)}) \leq 2^{-N} M^{-(d-|I|-2)/2}$  for all  $j \in I$  and  $k \in \{1, \dots, d-1\} \setminus I$ . Hence relation (7.3) also holds.

First we prove the existence of a partition with less than  $2^{C2^{2N}M}$  elements satisfying (7.1) only in the case  $|I| \leq d-3$ . This will be done with the help of the corollary of Proposition 4.2 when it is applied to the metric  $\bar{\rho}_{\alpha_{u^{(l)}}^I}$  and the norm  $\alpha_{u^{(l)}}^I$  inducing it. These quantities were introduced in (6.5) and (6.6). In the proof we need good estimates on the terms  $W_K^u(\alpha_{u^{(l)}}^I, t)$  defined in (4.2) for all sets  $K \subset \{1, \dots, d-1\} \setminus I$ ,  $K \neq \emptyset$  and  $u \in B^{n_{j_1}} \times \dots \times B^{n_{j_s}}$  with a number  $t$  chosen as  $t = c2^{-N}M^{-1/2}$  with a sufficiently small constant  $1 \geq c > 0$ . This quantity will be bounded by means of the estimates (7.2), (7.3) and Lemma 6.3. More precisely an equivalent version of Lemma 6.3 will be applied where  $B_{u^{(l)}}^I$  (defined in (6.2)) is chosen as the operator  $A$ , and as a consequence  $\alpha_{u^{(l)},s}(I)$  defined in (6.3) plays the role of the term  $\alpha_s = \alpha_s(A)$ . This term must satisfy relation (5.6) to have the right to apply Lemma 6.3. (Actually the variables of the operator  $B_{u^{(l)}}^I$  have to be reindexed if we want to apply Lemma 6.3 in its original form.) The operator  $B_{u^{(l)}}^I$  acts on the functions on  $\{1, \dots, d\} \setminus I$ , on a set of  $d - |I|$  elements, and by relation (7.2)  $\alpha_{u^{(l)},s}(I) \leq M^{-(s-1)/2}$  if  $1 \leq s \leq d - |I| - 1$ . This means that formula (5.6) holds for the operator we get by an appropriate reindexation the indices  $\{1, \dots, d\} \setminus I$  of the arguments of  $B_{u^{(l)}}^I$  to the set  $1, \dots, d - |I|$ . An appropriate reindexation is obtained if the elements of the set  $\{1, \dots, d\} \setminus I$  are listed in a monotone increasing order, and the  $j$ -th element of this sequence gets the index  $j$ . Such a reindexation of the indices yields a version of Lemma 6.3 that enables us to estimate the terms  $W_K^u(\alpha_{u^{(l)}}^I, t)$ . (Originally we get an estimate for a version of  $W_K^u(\alpha_{u^{(l)}}^I, t)$  with reindexed parameters by means of a version of  $B_{u^{(l)}}^I$  with reindexed parameters.)

In the application of Lemma 6.3 we still have to understand what  $\tilde{\alpha}_{j,k}(u_j)$  means in formula (6.9) if  $B_{u^{(l)}}^I$  plays the role of the operator  $A$ .

By formula (6.8) in Lemma 6.3 we get that

$$W_K^u(\alpha_{u^{(l)}}^I, t) \leq \frac{t^{|K|}}{M^{(d-|I|-|K|-1)/2}} \leq c^2 2^{-2N} M^{-(d-|I|-1)/2} \quad \text{if } 2 \leq |K| \leq d - |I| - 1.$$

I claim that relations (6.9) and (7.3) imply that

$$W_{\{k\}}^u(\alpha_{u^{(l)}}^I, t) \leq t \tilde{\alpha}_k^I(u^{(l)}) \leq c^2 2^{-2N} M^{-(d-|I|-1)/2}$$

for a one point set  $\{k\} \in \{1, \dots, d-1\} \setminus I$ . We get this bound from (7.3) if we show that  $\tilde{\alpha}_{j,k}(u_j) \leq \alpha_k^I(u^{(l)})$  for any  $j \in \{1, \dots, d-1\} \setminus I$ ,  $j \neq k$ , with the function  $\tilde{\alpha}_{j,k}(u_j)$  corresponding to the operator  $B_{u^{(l)}}^I$  if  $u_j \in B^{n_j}$ .

This inequality can be seen by giving a good representation of  $\tilde{\alpha}_{j,k}(u_j)$  when it corresponds to  $B_{u^{(l)}}^I$  instead of  $A$  together with a similar representation of  $\tilde{\alpha}_k^I(u^{(l)})$ . An adaptation of formula (5.4) will be applied to this case. The main difference between formula (5.4) and the representation of  $\tilde{\alpha}_{j,k}(u_j)$  given below is that in the new formula we have the fixed functions  $u_s^{(l)}(\cdot)$  in the coordinates  $s \in I$ . In this case we have

$$\begin{aligned} \tilde{\alpha}_{j,k}(u_j) = & \sup_{v_p(\cdot), p \in \{1, \dots, d-1\} \setminus (I \cup \{j, k\}), v_{k,d}(\cdot, \cdot)} \sum_{i_1, \dots, i_d} a(i_1, \dots, i_d) u_j(i_j) v_{k,d}(i_k, i_d) \\ & \prod_{s \in I} u_s^{(l)}(i_s) \prod_{p \in \{1, \dots, d-1\} \setminus (I \cup \{j, k\})} v_p(i_p) \end{aligned}$$

for a vector  $u_j \in B^{n_j}$ , where the supremum is taken for such vectors  $v_p(\cdot)$  depending on the coordinate  $i_p$ ,  $p \in \{1, \dots, d-1\} \setminus (I \cup \{j, k\})$ , for which  $\sum_{i_s} v_p^2(i_p) \leq 1$  and a function  $v_{k,d}(\cdot, \cdot)$ , depending on the coordinates  $i_k$  and  $i_d$  such that  $\sum_{i_k, i_d} v^2(i_k, i_d) \leq 1$ . The expression  $\tilde{\alpha}_k^I(u^{(l)})$  has a similar representation, only in its definition we have to take supremum also for all vectors  $v_j(\cdot) \in B^{n_j}$  in its  $j$ -th coordinate instead of fixing a vector  $u_j \in B^{n_j}$  as it was done in the definition of  $\tilde{\alpha}_{j,k}(u_j)$ ,  $u_j \in B^{n_j}$ . These observations imply the desired inequality  $\tilde{\alpha}_{j,k}(u_j) \leq \alpha_k^I(u^{(l)})$ .

The above inequalities imply that

$$\sum_{J: J \subset \{1, \dots, d-1\} \setminus I, J \neq \emptyset} W_J^u(\alpha_{u^{(l)}}^I, 4t) \leq 2^{-2N} M^{-(d-|I|-1)/2}$$

for all  $u \in B^{n_{j_1}} \times \dots \times B^{n_{j_s}}$  if the constant  $c > 0$  in the choice  $t = c2^{-N} M^{-1/2}$  is sufficiently small. Hence it follows from the corollary of Proposition 4.2 applied for the metric  $\bar{\rho}_{\alpha_{u^{(l)}}^I}$  induced by the norm  $\alpha_{u^{(l)}}^I$  with the choice  $D = B^{n_{j_1}} \times \dots \times B^{n_{j_r}}$  and  $t = c2^{-N} M^{-1/2}$  with a sufficiently small number  $c > 0$  and  $u = 2^{-2N} M^{-(d-|I|-1)/2}$  that relation (7.1) holds.

In the case  $|I| = d-2$  we can write  $I = \{1, \dots, d-1\} \setminus \{k\}$  with an appropriate  $k \in \{1, \dots, d-1\}$ . The inequality  $\tilde{\alpha}_{j,k}(u^{(l)}) \leq 2^{-N} M^{(d-2)/2}$  with an arbitrary index  $j \in I$  implies in this case that

$$\sum_{i_k, i_d} b_{u^{(l)}}^I(i_k, i_d) v(i_k, i_d) \leq M^{|I|/2} 2^{-N} M^{-(d-2)/2} \leq 2^{-N} \quad \text{if} \quad \sum_{i_k, i_d} v^2(i_k, i_d) \leq 1,$$

or in an equivalent form

$$\sum_{i_k, i_d} b_{u^{(l)}}^I(i_k, i_d)^2 \leq 2^{-2N}, \quad (7.4)$$

where  $I = \{1, \dots, d-1\} \setminus \{k\}$ , and the numbers  $b_{u^{(l)}}^I(i_k, i_d)$  are defined in (6.1). Let us also define the pseudonorm

$$\beta_{u^{(l)}}^I(v) = \left[ \sum_{i_d} \left( \sum_{i_k} b_{u^{(l)}}^I(i_k, i_d) v(i_k) \right)^2 \right]^{1/2}$$

of the vectors  $v = (v(1), \dots, v(n_k)) \in R^{n_k}$ .

The pseudometric  $\bar{\rho}_{\alpha_{u^{(l)}}^I}$  defined in (6.5) agrees in this case with the metric induced by the pseudonorm  $\beta_{u^{(l)}}^I$ . Hence in this case the existence of a partition  $V_1, \dots, V_L$  of  $B^{n_k}$  with  $L \leq 2^{C2^{2N}M}$  elements and the property  $\bar{\rho}_{\alpha_{u^{(l)}}^I}(v, \bar{v}) \leq 2^{-2N} M^{-1/2}$ , if  $v, \bar{v} \in V_l$  with some  $1 \leq l \leq L$ , i.e. relation (7.1) can be proved with the help of the corollary of Proposition 4.2 and the following estimate on the pseudonorm  $\beta_{u^{(l)}}^I$ .

By the Schwarz inequality and formula (7.4)

$$\begin{aligned}
E\beta_{u^{(l)}}^I(G_k) &\leq \left[ \sum_{i_d} E \left( \sum_{i_k} b_{u^{(l)}}^I(i_k, i_d) g_k(i_k) \right)^2 \right]^{1/2} \\
&= \left[ \sum_{i_k, i_d} b_{u^{(l)}}^I(i_k, i_d)^2 \right]^{1/2} \leq 2^{-N}
\end{aligned} \tag{7.5}$$

for a standard normal random vector  $G_k = (g_k(1), \dots, g_k(n_k))$  of dimension  $n_k$ . Because of relation (7.5) an application of the corollary of Proposition 4.2 for the operator  $\beta_{u^{(l)}}^I$  with  $4t = 2^{-(N+1)}M^{-1/2}$  and  $u = 2^{-2N}M^{-1/2}$  shows the existence of a partition  $V_1, \dots, V_L$  of  $B^{n_k}$  with  $L \leq 2^{C_1/t^2} \leq 2^{C2^{2N}M}$  elements such that  $\bar{\rho}_{\alpha_{u^{(l)}}^I}(v, \bar{v}) = \beta_{u^{(l)}}^I(v - \bar{v}) \leq 2^{-2N}M^{-1/2}$  if  $v \in V_l, \bar{v} \in V_l$  with some  $1 \leq l \leq L$ . We had to prove this statement.

Let us fix some  $u^{(l)}$  appearing in the partition  $u^{(l)} + U_l$ ,  $1 \leq l \leq L$  of the set  $U$  we are considering. It can be shown with the help of relation (7.1) that there exists a partition  $V_1(l), \dots, V_{L_l}(l)$  of  $B^{n_1} \times \dots \times B^{n_{d-1}}$  with  $L_l \leq 2^{C2^{2N}M}$  elements such that

$$\begin{aligned}
\rho_{\alpha_{u^{(l)}}^I}(v, \bar{v}) &\leq 2^{-2N}M^{-(d-|I|-1)/2} \quad \text{if } v \in V_p(l) \text{ and } \bar{v} \in V_p(l) \text{ with the same} \\
&\text{index } p \text{ for all } I \subset \{1, \dots, d-1\} \text{ such that } 1 \leq |I| \leq d-2.
\end{aligned} \tag{7.6}$$

Indeed, it follows from (7.1) and the definition of  $\rho_{\alpha_{u^{(l)}}^I}$  in (6.5) and (6.7) that for all sets  $I \subset \{1, \dots, d-1\}$ ,  $1 \leq |I| \leq d-2$ , there is a partition  $V_1(l, I), \dots, V_{L(l, I)}(l, I)$  of  $B^{n_1} \times \dots \times B^{n_{d-1}}$  depending on  $l$  and  $I$  with  $L(l, I) \leq 2^{C2^{2N}M}$  elements such that  $\rho_{\alpha_{u^{(l)}}^I}(v, \bar{v}) \leq 2^{-2N}M^{-(d-|I|-1)/2}$  if  $v \in V_p(l, I)$  and  $\bar{v} \in V_p(l, I)$  with the same index  $p$ . Then taking all possible intersections  $\bigcap_{I: I \subset \{1, \dots, d-1\}, 1 \leq |I| \leq d-2} V_{p(I)}(I, l)$

that contain exactly one element from each above introduced partitions depending on the sets  $I \subset \{1, \dots, d-1\}$ ,  $1 \leq |I| \leq d-2$ , we get a partition of  $B^{n_1} \times \dots \times B^{n_{d-1}}$  that satisfies (7.6). Let us observe that the number of elements of this partition also can be bounded from above by  $2^{C2^{2N}M}$  with some constant  $C > 0$ .

Let us choose a partition  $V_1(l), \dots, V_{L_l}(l)$  of  $B^{n_1} \times \dots \times B^{n_{d-1}}$  satisfying relation (7.6) for all vectors  $u^{(l)}$  taking part in a partition  $u^{(l)} + U_l$ ,  $1 \leq l \leq L$  satisfying Lemma 6.1. Then the ensemble of sets  $u^{(l, p)} + (V_p(l) \cap U_l)$ ,  $1 \leq p \leq L(l)$ ,  $1 \leq l \leq L$ , with  $u^{(l, p)} = u^{(l)}$  constitutes a partition of the set  $U$  which, after an appropriate reindexation, satisfies Lemma 6.2.

## 8. The proof of the Main inequality.

In this section I prove the Main inequality with the help of Lemma 6.2.

*The proof of the Main inequality.* First it will be shown that relation (5.7) holds with an appropriate constant  $C = C(d)$  in it if  $N \geq N_0$  with a sufficiently large threshold index  $N_0 = N_0(r)$ . To this end let us observe that

$$E(Y_d(u) - Y_d(u'))^2 = \rho_\alpha(u, u')^2,$$

hence

$$E(Y_d(u) - Y_d(u'))^{2M} = 1 \cdot 3 \cdot \dots \cdot (2M - 1) \rho_\alpha(u, u')^{2M} \leq (2M)^M \rho_\alpha(u, u')^{2M}$$

with the metric  $\rho_\alpha$  defined in (2.11) for arbitrary vectors  $u \in R^{n_1} \times \dots \times R^{n_{d-1}}$ ,  $u' \in R^{n_1} \times \dots \times R^{n_{d-1}}$  and  $M \geq 1$ . In particular,

$$\begin{aligned} E(Y_d(u^{(t)}) - Y_d(u^{(t')}))^{2^{2(N+A)}M} &\leq (2^{2(N+A)}M)^{2^{2(N+A)}M/2} \cdot (2^{-2N}M^{-(d-1)/2})^{2^{2(N+A)}M} \\ &= (2^{(A-N)}M^{-(d-2)/2})^{2^{2(N+A)}M} \end{aligned} \quad (8.1)$$

for all  $u^{(t)} \in U$  and  $u^{(t')} \in U$  if  $U \in \mathcal{U}(r, N)$ . As a consequence,

$$\begin{aligned} E \left[ \sup_{(u^{(t)}, u^{(t')}) : u^{(t)} \in U, u^{(t')} \in U} (Y_d(u^{(t)}) - Y_d(u^{(t')})) \right]^{2^{2(N+A)}M} \\ \leq r^2 (M^{-(d-2)/2} 2^{(A-N)})^{2^{2(N+A)}M} \leq (2M^{-(d-2)/2} \cdot 2^{(A-N)})^{2^{2(N+A)}M} \\ \leq (CM^{-(d-2)/2} 2^{(A-N)})^{2^{2(N+A)}M} \end{aligned}$$

if  $N \geq N_0(r)$  with some threshold  $N_0(r)$  and constant  $C \geq 2$ , i.e. relation (5.7) holds for  $N \geq N_0$  with  $C = C(d) \geq 2$  and  $A \geq A_0 \geq 0$ .

Hence it is enough to show that relation (5.7) holds for a set  $U \in \mathcal{U}(r, N)$  if it holds for all sets  $U \in \mathcal{U}(r, N + 2)$ . To show this let us consider such a partition  $u^{(l)} + U_l$ ,  $1 \leq l \leq L$  of the set  $U \in \mathcal{U}(r, N)$  with  $L \leq 2^{C2^{2N}M}$  elements which satisfies Lemma 6.2. First the following weaker estimate will be verified.

Let us take an element  $u^{(l)} + U_l$  of the partition of  $U$  we consider. Let us denote this set by  $\bar{U}_l$ . We will show that the estimate

$$\begin{aligned} E \left[ \sup_{(u^{(t)}, u^{(t')}) : u^{(t)} \in \bar{U}_l, u^{(t')} \in \bar{U}_l} (Y_d(u^{(t)}) - Y_d(u^{(t')})) \right]^{2^{2(N+A)}M} \\ \leq \left( \frac{C}{3} M^{-(d-2)/2} 2^{(A-N)} \right)^{2^{2(N+A)}M} \end{aligned} \quad (8.2)$$

holds for all  $A \geq A_0$  with some threshold index  $A_0$  and the same constant  $C = C(d)$  which appears in (5.7) (with parameter  $N+2$ ) if these constant (depending only on the parameter  $d$ ) are chosen sufficiently large.

To prove relation (8.2) let us consider two arbitrary vectors  $u \in \bar{U}_l$  and  $u' \in \bar{U}_l$ , write them in the form  $u = u^{(l)} + u^{(0)}$  and  $u' = u^{(l)} + u'^{(0)}$  with  $u^{(0)} \in U_l$  and  $u'^{(0)} \in U_l$ . We can write the difference  $Y_d(u) - Y_d(u')$  because of the special form of relation (2.1) defining  $Y_d(u)$  as

$$\begin{aligned} Y_d(u) - Y_d(u') &= Y_d(u^{(l)} + u^{(0)}) - Y_d(u^{(l)} + u'^{(0)}) = Y_d(u^{(0)}) - Y_d(u'^{(0)}) \\ &+ \sum_{I: I \subset \{1, \dots, d-1\}, 1 \leq |I| \leq d-2} M^{-|I|/2} \left[ Y_{u^{(l)}}^I(u^{(0)}) - Y_{u^{(l)}}^I(u'^{(0)}) \right], \end{aligned} \quad (8.3)$$

where

$$Y_{u^{(l)}}^I(v) = \sum_{(i_j, j \in \{1, \dots, d-1\} \setminus I)} b_{u^{(l)}}^I(i_j, j \in \{1, \dots, d\} \setminus I) \prod_{j \in \{1, \dots, d-1\} \setminus I} v_j(i_j) g_d(i_d)$$

for all  $v = (v_1(i_1), \dots, v_{d-1}(i_{d-1})) \in R^{n_1} \times \dots \times R^{n_{d-1}}$  and  $I \subset \{1, \dots, d-1\}$ ,  $1 \leq |I| \leq d-2$  with the constants  $b_{u^{(l)}}^I(i_j, j \in \{1, \dots, d\} \setminus I)$  defined in (6.1). (Here we apply this formula with the choice  $u = u^{(l)}$  and  $(g_d(1), \dots, g_d(n_d))$  is the same vector of independent, standard Gaussian random variables which appeared in the definition of  $Y_d(u)$ .)

In the subsequent considerations the following notation will be applied. Given some vector  $u^{(t)} \in \bar{U}_l$ , its decomposition to the vector  $u^{(l)}$  plus a vector in  $U_l$  will be denoted as  $u^{(t)} = u^{(l)} + u^{(t,0)}$  with  $u^{(t,0)} \in U_l$ .

By taking the supremum of the expressions both at the left-hand and right-hand side of identity (8.3) for all pairs  $(u^{(t)}, u^{(t')})$  such that  $u^{(t)} \in \bar{U}_l$  and  $u^{(t')} \in \bar{U}_l$  we get an identity that implies the following inequality.

$$\sup_{(u^{(t)}, u^{(t')}) : u^{(t)} \in \bar{U}_l, u^{(t')} \in \bar{U}_l} (Y_d(u^{(t)}) - Y_d(u^{(t')})) \leq Z + \sum_{I: I \subset \{1, \dots, d-1\}, 1 \leq |I| \leq d-2} M^{-|I|/2} Z_I \quad (8.4)$$

with

$$Z = Z(l, N) = \sup_{(u^{(t,0)}, u^{(t',0)}) : u^{(t)} \in U_l, u^{(t',0)} \in U_l} (Y_d(u^{(t,0)}) - Y_d(u^{(t',0)}))$$

and

$$\begin{aligned} Z_I &= Z_I(l, N) = \sup_{(u^{(t,0)}, u^{(t',0)}) : u^{(t)} \in U_l, u^{(t',0)} \in U_l} [Y_{u^{(l)}}^I(u^{(t,0)}) - Y_{u^{(l)}}^I(u^{(t',0)})], \\ &\text{for all } I \subset \{1, \dots, d-1\} \text{ such that } 1 \leq |I| \leq d-2. \end{aligned}$$

I claim that

$$EZ^{2^{(N+A)}M} \leq \left( \frac{C}{4} M^{-(d-2)/2} 2^{(A-N)} \right)^{2^{2(N+A)}M} \quad (8.5)$$

with the same constant  $C = C(d)$  as in formula (5.7) if  $A \geq A_0$  with some fixed number  $A_0 = A_0(d) \geq 0$ , and

$$EZ_I^{2^{(N+A)}M} \leq (C'M^{-(d-|I|-2)/2}2^{(A-N)})^{2^{(N+A)}M} \quad (8.6)$$

for all  $I \subset \{1, \dots, d-1\}$  such that  $1 \leq |I| \leq d-2$  if  $A \geq A_0$  with some universal constants  $A_0$  and  $C'$ . Let me emphasize in particular that the constants  $A_0$  and  $C'$  in (8.6) do not depend on the choice of the constant  $C = C(d)$  and threshold index  $A_0$  in (5.7).

Relation (8.5) can be deduced from our inductive hypothesis by which the Main inequality holds for  $N+2$  and the fact that  $U_l \in \mathcal{U}(r, N+2)$ . Indeed, this inductive hypothesis together with Hölder's inequality yield that

$$\begin{aligned} EZ^{2^{(N+A)}M} &= E \left[ \sup_{((u^{(t,0)}, u^{(t',0)})) : u^{(t,0)} \in U_l, u^{(t',0)} \in U_l} (Y_d(u^{(t,0)}) - Y_d(u^{(t',0)})) \right]^{2^{(N+A)}M} \\ &\leq \left[ E \left[ \sup_{((u^{(t,0)}, u^{(t',0)})) : u^{(t,0)} \in U_l, u^{(t',0)} \in U_l} (Y_d(u^{(t,0)}) - Y_d(u^{(t',0)})) \right]^{2^{(N+A+2)}M} \right]^{1/4} \\ &\leq \left( CM^{-(d-2)/2}2^{(A-(N+2))} \right)^{2^{(N+A+2)}M/4} = \left( \frac{1}{4} CM^{-(d-2)/2}2^{(A-N)} \right)^{2^{(N+A)}M}. \end{aligned}$$

(The reason for applying the induction from  $N+2$  and not from  $N+1$  to  $N$  in our proof is that in such a way we got a coefficient  $\frac{1}{4}$  at the right-hand side of estimate (8.5). An induction from  $N+1$  to  $N$  would yield only a weaker estimate with multiplying factor  $\frac{1}{2}$  which would be not sufficient for our purposes.)

Relation (8.6) will be proved first only in the case  $1 \leq |I| \leq d-3$ . This will be done with the help of the Main inequality with parameter  $d-|I| \leq d-1$ . This is legitime because of our inductive hypothesis. The main inequality will be applied for the operator  $B_{u^{(I)}}^I$  defined in (6.2) as the operator  $A$  and the set of vectors  $U \in \mathcal{U}(r, N)$  will be chosen as  $U = U_l(I) = \{u_{I^c}^{(t,0)} : u^{(t,0)} \in U_l\}$ . That is we get the set  $U$  by taking the vectors  $u = (u_1, \dots, u_{d-1}) \in U_l$  and omitting their coordinates indexed by the elements of the set  $I$ . More precisely, we apply the Main inequality for a version of  $B_{u^{(I)}}^I$  and  $U_l(I)$  we get by renummerating the indices of their coordinates to the sets  $\{1, \dots, d-|I|\}$  and  $\{1, \dots, d-|I|-1\}$  respectively in an appropriate way. A good way of reindexation of the coordinates is to list them with monotone increasing indices and to give then the  $j$ -th element the index  $j$ .

To apply the Main inequality we have to show that its conditions are satisfied with such a choice. We have to check that the operator  $B_{u^{(I)}}^I$  satisfies relation (5.6). (Here  $d-|I|$  takes the role of the parameter  $d$ .) This statement follows from the analogous statement for the operator  $A$ . Beside this, we have to show that  $U_l(I) \in \mathcal{U}_{B_{u^{(I)}}^I, M, d-|I|}^I(r, N)$ . This can be done with the help of Lemma 6.2.



The estimate we have to give on  $\bar{\rho}_{\alpha_{u^{(l)}}^I}(\cdot, \cdot)$  to show that  $U_l(I) \in \mathcal{U}(r, N) = \mathcal{U}_{B_{u^{(l)}}^I, M, d-|I|}(r, N)$  agrees with the estimate we proved in Lemma 6.2 on this quantity. The bound we have to give about  $\tilde{\alpha}_{j,k}$  to show that  $U_l(I) \in \mathcal{U}_{B_{u^{(l)}}^I, M, d-|I|}(r, N)$  follows from relation (7.3) and the inequality  $\tilde{\alpha}_{j,k}(u_j) \leq \tilde{\alpha}_k^I(u^{(l)})$  if  $u_j \in B^{n_j}$  with the same quantities  $\tilde{\alpha}_{j,k}(u_j)$  and  $\tilde{\alpha}_k^I(u^{(l)})$  which appeared in the proof of Lemma 6.2. The remaining properties needed to check that  $U_l(I) \in \mathcal{U}_{B_{u^{(l)}}^I, M, d-|I|}(r, N)$  clearly hold. Then the Main inequality may be applied with such a choice, and it yields relation (8.6) for  $1 \leq |I| \leq d-3$ .

In the case  $|I| = d-2$ , and the set  $\{1, \dots, d-1\} \setminus I$  consists of a point  $k$ , and formula (8.6) can be proved with the help of the Main inequality in the case  $d=2$  in a similar way. This inequality can be applied for the operator  $2^N B_{u^{(l)}}^I$  defined as  $2^N B_{u^{(l)}}^I(v) = \sum_{i_k, i_d} 2^N b_{u^{(l)}}^I(i_k, i_d)v(i_k, i_d)$  for a vector  $v \in R^{n_k} \otimes R^{n_d}$  as the operator  $A$ . It follows from Lemma 6.2 that

$$\alpha_2(u^{(t,0)} - u^{(t',0)}) = \left[ \sum_{i_d} \left( \sum_{i_k} 2^N b_{u^{(l)}}^I(i_k, i_d) [u_{I^c}^{(t,0)}(i_k) - u_{I^c}^{(t',0)}(i_k)] \right)^2 \right]^{1/2} \leq 2^{-N} M^{-1/2}$$

if  $u^{(t,0)}(i_k) \in U_l$  and  $u^{(t',0)}(i_k) \in U_l$ . Hence the set consisting of all vectors of the form  $\frac{1}{2}(u_{I^c}^{(t,0)} - u_{I^c}^{(t',0)})$  with some  $u^{(t,0)} \in U_l$  and  $u^{(t',0)} \in U_l$  is contained in the set  $U_N$  introduced in (3.2). The inequality  $\alpha_1(2^N B_{u^{(l)}}^I) \leq 1$  also holds by formula (7.4). Hence the Main inequality in the case  $d=2$  can be applied in this case, and it yields that  $E(2^N Z_I)^{2^{2(N+A)}M} \leq (C \cdot 2^A)^{2^{2(N+A)}M}$  which is equivalent to (8.6) for  $|I| = d-2$ .

Inequality (8.2) follows from relations (8.4), (8.5), (8.6) and Minkowski's inequality for  $L_p$  norms with  $p = 2^{2(N+A)}M$ . (Observe that we are working with non-negative random variables, since the supremums we consider contain the terms  $Y_d(u^{(t)}) - Y_d(u^{(t')}) \equiv 0$ .) Indeed, they yield that

$$\begin{aligned} E \left[ \sup_{(u^{(t)}, u^{(t')}) : u^{(t)} \in \bar{U}_l, u^{(t')} \in \bar{U}_l} (Y_d(u^{(t)}) - Y_d(u^{(t')})) \right]^{2^{2(N+A)}M} \\ \leq \left( \left( \frac{C}{4} + 2^d C' \right) M^{-(d-2)/2} 2^{(A-N)} \right)^{2^{2(N+A)}M}. \end{aligned}$$

If the constant  $C = C(d)$  in the Main inequality is chosen sufficiently large, then  $\frac{C}{4} + 2^d C' \leq \frac{C}{3}$  in the last inequality, and this means that it implies relation (8.2).

The Main inequality will be proved with the help of inequality (8.2). It will be also exploited that the cardinality of the partition of a set  $U$  in Lemma 6.2 is not too large.

Let us consider a partition  $\bar{U}_l = u^{(l)} + U_l$ ,  $1 \leq l \leq L$ , of a set  $U \in \mathcal{U}(r, N)$  with  $L \leq 2^{C2^{2N}M}$  elements that satisfies Lemma 6.2. Let us fix an element  $\bar{u}^{(l)} \in \bar{U}_l$  in all

sets  $\bar{U}_l$ ,  $1 \leq l \leq L$ . Given a vector  $u^{(t)} \in U$  let  $\ell(t)$  denote that index  $l$ ,  $1 \leq l \leq L$ , for which  $u^{(t)} \in \bar{U}_l$ . Then we can write for two arbitrary vectors  $u^{(t)} \in U$  and  $u^{(t')} \in U$  the inequality

$$\begin{aligned} Y_d(u^{(t)}) - Y_d(u^{(t')}) &= [Y_d(u^{(t)}) - Y_d(\bar{u}^{\ell(t)})] + [Y_d(u^{(t')}) - Y_d(\bar{u}^{\ell(t')})] \\ &\quad + [Y_d(\bar{u}^{\ell(t')}) - Y_d(\bar{u}^{\ell(t)})] \\ &\leq 2 \sup_{1 \leq l \leq L} \sup_{u^{(s)}: u^{(s)} \in \bar{U}_l} [Y_d(u^{(s)}) - Y_d(\bar{u}^{(l)})] + \sup_{1 \leq l, l' \leq L} [Y_d(\bar{u}^{(l')}) - Y_d(\bar{u}^{(l)})]. \end{aligned}$$

Since the right-hand side of the above inequality does not depend on the vectors  $u^{(t)} \in U$  and  $u^{(t')} \in U$  it implies that

$$\begin{aligned} &\sup_{(u^{(t)}, u^{(t')}) : u^{(t)} \in U, u^{(t')} \in U} [Y_d(u^{(t)}) - Y_d(u^{(t')})] \\ &\leq 2 \sup_{1 \leq l \leq L} \sup_{u^{(s)}: u^{(s)} \in \bar{U}_l} [Y_d(u^{(s)}) - Y_d(\bar{u}^{(l)})] + \sup_{1 \leq l, l' \leq L} [Y_d(\bar{u}^{(l')}) - Y_d(\bar{u}^{(l)})]. \end{aligned} \quad (8.7)$$

The Main inequality can be proved by means of good moment estimates on the two terms at the right-hand side of inequality (8.7). It follows from inequalities (8.2) and  $L \leq 2^{C'2^{2N}M}$  for the number of partitions of  $U$  in Lemma 6.2, where the number  $C'$  does not depend on the constants  $A_0 = A_0(d)$  and  $C = C(d)$  in the Main inequality that

$$\begin{aligned} &E \left( 2 \sup_{1 \leq l \leq L} \sup_{u^{(s)}: u^{(s)} \in \bar{U}_l} [Y_d(u^{(s)}) - Y_d(\bar{u}^{(l)})] \right)^{2^{2(N+A)}M} \\ &\leq \sum_{l=1}^L E \left( 2 \sup_{(u^{(s)}, u^{(s')}) : u^{(s)} \in \bar{U}_l, u^{(s')} \in U_l} [Y_d(u^{(s)}) - Y_d(\bar{u}^{(s')})] \right)^{2^{2(N+A)}M} \\ &\leq L \left( \frac{2C}{3} M^{-(d-2)/2} 2^{(A-N)} \right)^{2^{2(N+A)}M} \\ &\leq 2^{C'2^{2N}M} \left( \frac{2C}{3} M^{-(d-2)/2} 2^{(A-N)} \right)^{2^{2(N+A)}M} \\ &= \left( 2^{C'2^{-2A}} \frac{2C}{3} M^{-(d-2)/2} 2^{(A-N)} \right)^{2^{2(N+A)}M} \leq \left( \frac{3C}{4} M^{-(d-2)/2} 2^{(A-N)} \right)^{2^{2(N+A)}M} \end{aligned} \quad (8.8)$$

if the threshold index  $A_0$  in the Main inequality is chosen so large that  $2^{C'2^{-2A}} \leq \frac{9}{8}$  for  $A \geq A_0$ . Such a choice is possible since the constant  $C'$  appearing in the exponent of the bound for the cardinality of the partition of the set  $U$  does not depend on the choice of the number  $A_0$  in the Main inequality. (The threshold index  $A_0$  was introduced to have a control on the multiplicative factor  $L$  in the previous estimate.)

We get in a similar way with the help of inequality (8.1)

$$\begin{aligned}
E \left( \sup_{1 \leq l, l' \leq L} [Y_d(\bar{u}^{(l')}) - Y_d(\bar{u}^{(l)})] \right)^{2^{2(N+A)}M} &\leq \sum_{1 \leq l, l' \leq L} E \left( [Y_d(\bar{u}^{(l')}) - Y_d(\bar{u}^{(l)})] \right)^{2^{2(N+A)}M} \\
&\leq L^2 \left( M^{-(d-2)/2} 2^{(A-N)} \right)^{2^{2(N+A)}M} \leq 2^{2C'2^{2N}M} \left( M^{-(d-2)/2} 2^{(A-N)} \right)^{2^{2(N+A)}M} \\
&\leq \left( \bar{C} M^{-(d-2)/2} 2^{(A-N)} \right)^{2^{2(N+A)}M} \tag{8.9}
\end{aligned}$$

with a constant  $\bar{C}$  which does not depend on the constant  $C = C(d)$  in the Main inequality.

It follows from relations (8.7), (8.8), (8.9) and Minkowski's inequality for  $L_p$  norms with  $p = 2^{2(N+A)}M$  (we are working again with non-negative random variables) that

$$\begin{aligned}
E \left[ \sup_{(u^{(t)}, u^{(t')}) : u^{(t)} \in U, u^{(t')} \in U} (Y_d(u^{(t)}) - Y_d(u^{(t')})) \right]^{2^{2(N+A)}M} \\
\leq \left( \left( \frac{3C}{4} + \bar{C} \right) M^{-(d-2)/2} 2^{(A-N)} \right)^{2^{2(N+A)}M} \leq (C M^{-(d-2)/2} 2^{(A-N)})^{2^{2(N+A)}M}
\end{aligned}$$

if the constant  $C = C(d)$  (together with  $A_0 = A_0(d)$ ) is chosen sufficiently large. The Main inequality is proved.

*Remark.* In the proof of the Main inequality Lemma 6.2 played an important role. In the verification of relation (8.2) we have exploited that the partition of the set  $U$  we have considered satisfies all properties formulated in Lemma 6.2. Latała tried to prove an analogous estimate for all partitions satisfying Lemma 6.1. His proof however contains an error. It applies an estimate formulated in relation (18) of Lemma 7 in paper [2] whose verification is based on Theorem 3 of [2]. But the proof of this Theorem 3 is incorrect. This result was proved by means of a backward induction similarly to our Main inequality. In Latała's backward induction procedure we turn from the parameters  $(r-1, l+1)$  to  $(r, l)$ . (In this remark I apply the notation of paper [2].) But the last step of this backward induction when we turn from  $l=1$  to  $l=0$  does not work.

The essential statement of Theorem 3 in [2] proved by backward induction was formulated by means of some objects denoted by  $\Delta_l$  and  $\tilde{\Delta}_l$ . These objects were defined differently for  $l=0$  and  $l \geq 1$ . Hence a special argument would have been needed to prove the induction step (formulated in relation (25) of [2]) for  $l=0$ . But such a step is missing from the proof of [2]. Moreover, it would require different arguments. The situation seems to be similar to the proof of the Basic estimate with the help of the Main inequality in this paper where formula (5.6) had to be replaced by the stronger condition (1.10). I think that in the last step of the backward induction proof of Theorem 3 in [2] such a bound should appear which also depends on the term  $\|A\|_{\{1\}, \dots, \{d\}}$ , and this would supply only a weaker estimate.

This seems to be a serious error. I believe that not only the proof of Theorem 3 is incorrect, but even the results formulated in Theorem 3 and relation (18) of [2] are wrong. Since Latała's proof heavily exploited formula (18) it was not clear for me whether his main result holds in its original form or it must be modified. My main goal in this paper was to answer this question. Finally it turned out that Latała's result is correct. But to prove this I had to find a new, better partition of the sets  $U \in \mathcal{U}(r, N)$  than Latała did. It was the partition constructed in Lemma 6.2 that helped in saving Latała's proof.

## Appendix. The proof of Propositions 4.1 and 4.2.

*The proof of Proposition 4.1.* Put  $K = \{y: y \in R^n, \alpha_1(y) \leq 4\alpha_1(tG), \alpha_2(y) \leq 4\alpha_2(tG)\}$ . Then  $\mu_{n,t}(K) \geq \frac{1}{2}$ , since by the Markov inequality

$$1 - \mu_{n,t}(K) \leq \mu_{n,t}(y: \alpha_1(y) > 4tE\alpha_1(tG)) + \mu_{n,t}(y: \alpha_2(y) > 4tE\alpha_2(tG)) \leq \frac{1}{2}.$$

Beside this, the set  $K$  has the symmetry property  $-K = K$  which yields that

$$\begin{aligned} \mu_{n,t}(y: y \in R^n, \alpha_1(y-x) \leq 4E\alpha_1(tG), \alpha_2(y-x) \leq 4E\alpha_2(tG)) \\ = C_n \int_{K+x} e^{-y^2/2t} dy = C_n \int_K e^{-(y+x)^2/2t} dy = e^{-\|x\|^2/2t} \int_K e^{(y,x)/t} \mu_{n,t}(dy) \\ = e^{-\|x\|^2/2t} \int_K \frac{1}{2} \left( e^{(y,x)/t} + e^{(-y,x)/t} \right) \mu_{n,t}(dy) \geq e^{-\|x\|^2/2t} \mu_{n,t}(K) \end{aligned}$$

with the norming constant  $C_n = (\sqrt{2\pi t})^{-n}$ . Hence the relations  $\mu_{n,t}(K) \geq \frac{1}{2}$ , and  $\|x\| \leq 1$  (i.e.  $x \in B^n$ ) imply that

$$\begin{aligned} \mu_{n,t}(\{y: y \in R^n, \alpha_1(y-x) \leq 4E\alpha_1(tG), \alpha_2(y-x) \leq 4E\alpha_2(tG)\}) \\ \geq \frac{1}{2} e^{-\|x\|^2/2t} \geq \frac{1}{2} e^{-1/2t^2}. \end{aligned}$$

*The proof of Proposition 4.2.* In the case  $d = 1$  Proposition 4.2 immediately follows from Proposition 4.1 if it is applied for  $\alpha = \alpha_1 = \alpha_2$ , and the relation  $4E\alpha(tG_n) = E\alpha(4tG_n)$  is exploited. Hence it is enough to prove Proposition 4.2 for  $d$  under the inductive hypothesis that it holds for  $d-1$ .

Let us fix some  $x = (x_1, \dots, x_d) \in B^{n_1} \times \dots \times B^{n_d}$ , where  $B^n$  denotes the unit ball in  $R^n$ . We can write

$$\begin{aligned} \rho_\alpha(x, y) &= \alpha(y_1 \otimes \dots \otimes y_{d-1} \otimes y_d - x_1 \otimes \dots \otimes x_{d-1} \otimes x_d) \\ &\leq \alpha(x_1 \otimes \dots \otimes x_{d-1} \otimes (y_d - x_d)) + \alpha((y_1 \otimes \dots \otimes y_{d-1} - x_1 \otimes \dots \otimes x_{d-1}) \otimes y_d) \end{aligned} \tag{A1}$$

for arbitrary  $y = (y_1, \dots, y_d) \in R^{n_1} \times \dots \times R^{n_d}$ .

We shall define some sets  $A$ ,  $B$  and  $C$ . We shall not denote their dependence on the vector  $x \in B^{n_1} \times \cdots \times B^{n_d}$  we have fixed. To define the set  $A$  first we introduce the following quantity  $W_I^x(y|\alpha, t)$  similar to the quantity  $W_I^x(\alpha, t)$  defined in formula (4.2).

Let us fix  $d$  independent standard normal vectors  $G_j = (g_j(1), \dots, g_j(n_j))$  of dimension  $n_j$ ,  $1 \leq j \leq d$ , and define for all  $t > 0$ ,  $y \in R^{n_d}$  and  $I \subset \{1, \dots, d\}$ ,  $I \neq \emptyset$  the quantity

$$W_I^x(y|\alpha, t) = E\alpha(z_1 \otimes \cdots \otimes z_d) \quad \text{where } z_j = tG_j \text{ if } j \in I, \\ z_j = x_j \text{ if } j \notin I \text{ and } j \neq d, \quad \text{and } z_d = y \text{ if } d \notin I.$$

Then we put

$$A = \left\{ y_d: y_d \in R^{n_d}, \alpha(x_1 \otimes \cdots \otimes x_{d-1} \otimes (y_d - x_d)) \leq E\alpha(x_1 \otimes \cdots \otimes x_{d-1} \otimes 4tG_d), \right. \\ \left. \sum_{I: I \subset \{1, \dots, d-1\}, I \neq \emptyset} W_I^x(y_d - x_d|\alpha, 4t) \leq \sum_{I: I \subset \{1, \dots, d\}, d \in I, I \cap \{1, \dots, d-1\} \neq \emptyset} W_I^x(\alpha, 4t) \right\},$$

$$B = \left\{ y = (y_1, \dots, y_d): y \in R^{n_1} \times \cdots \times R^{n_d}, \right. \\ \left. \alpha((y_1 \otimes \cdots \otimes y_{d-1} - x_1 \otimes \cdots \otimes x_{d-1}) \otimes y_d) \leq \sum_{I: I \subset \{1, \dots, d-1\}, I \neq \emptyset} W_I^x(y_d|\alpha, 4t) \right\}$$

and

$$C = B \cap \{y = (y_1, \dots, y_d): y \in R^{n_1} \times \cdots \times R^{n_d}, y_d \in A\}.$$

I claim that the inequalities

$$\mu_{n_d, t}(A) \geq \frac{1}{2} e^{-1/2t^2} \tag{A2}$$

and

$$\mu_{n_1 + \dots + n_{d-1}, t}(B \cap ((R^{n_1} \times \cdots \times R^{n_{d-1}}) \times y_d)) \geq 2^{-(d-1)} e^{-(d-1)/2t^2} \\ \text{for all } y_d = (y_d(1), \dots, y_d(n_d)) \in R^{n_d} \tag{A3}$$

hold, where  $(R^{n_1} \times \cdots \times R^{n_{d-1}}) \times y_d = \{(y_1, \dots, y_{d-1}, y_d): (y_1, \dots, y_{d-1}) \in R^{n_1} \times \cdots \times R^{n_{d-1}}\}$ .

Relation (A2) follows from the identity  $4E\alpha(tG_n) = E\alpha(4tG_n)$  and Proposition 4.1 with the choice  $\alpha_1(z) = \alpha(x_1 \otimes \cdots \otimes x_{d-1} \otimes z)$  and  $\alpha_2(z) = \sum_{I: I \subset \{1, \dots, d-1\}, I \neq \emptyset} W_I^x(z|\alpha, 4t)$  for  $z \in R^{n_d}$ . Observe that both  $\alpha_1(\cdot)$  and  $\alpha_2(\cdot)$  are pseudonorms in  $R^{n_d}$ , hence Proposition (4.1) is applicable for them.

Relation (A3) follows from Proposition 4.2 with parameter  $d-1$  if it is applied for the pseudonorm  $\bar{\alpha}_{y_d}$  on  $R^{n_1} \otimes \cdots \otimes R^{n_{d-1}}$  defined by the formula  $\bar{\alpha}_{y_d}(u) = \alpha(u \otimes y_d)$

for  $u \in R^{n_1} \otimes \dots \otimes R^{n_{d-1}}$  with a fixed  $y_d \in R^{n_d}$ . Here  $u \otimes y_d$  is that function in  $R^{n_1} \otimes \dots \otimes R^{n_d}$  for which  $u \otimes y_d(i_1, \dots, i_d) = u(i_1, \dots, i_{d-1})y_d(i_d)$  for all  $1 \leq i_j \leq n_j$ ,  $1 \leq j \leq d$ . Observe that  $\bar{\alpha}_{u_d}$  is really a pseudonorm for all  $y_d \in R^{n_d}$ , hence we can apply Proposition 4.2 with parameter  $d-1$  for it.

Relations (A2), (A3) and the Fubini theorem imply that

$$\mu_{n_1+\dots+n_d,t}(C) \geq 2^{-d}e^{-d/2t^2}. \quad (\text{A4})$$

Indeed,  $\mu_{n_1+\dots+n_{d-1},t}(B \cap ((R^{n_1} \times \dots \times R^{n_{d-1}}) \times y_d)) \geq 2^{-(d-1)}e^{-(d-1)/2t^2}$  for all points  $y_d \in R^{n_d}$  by relation (A3). We get relation (A4) from this inequality, relations (A2) and the Fubini theorem by integrating this inequality on the set  $\{y_d \in A\}$  with respect to the measure  $\mu_{n_d,t}$ .

Finally, I claim that

$$C \subset \left\{ y = (y_1, \dots, y_d): y \in R^{n_1} \times \dots \times R^{n_d}, \rho_\alpha(x, y) \leq \sum_{I: I \subset \{1, \dots, d\}, I \neq \emptyset} W_I^x(\alpha, 4t) \right\}. \quad (\text{A5})$$

Indeed, if  $y = (y_1, \dots, y_d) \in C$ , then

$$\begin{aligned} \rho_\alpha(x, y) &\leq \alpha(x_1 \otimes \dots \otimes x_{d-1} \otimes (y_d - x_d)) + \alpha((y_1 \otimes \dots \otimes y_{d-1} - x_1 \otimes \dots \otimes x_{d-1}) \otimes y_d) \\ &\leq E\alpha(x_1 \otimes \dots \otimes x_{d-1} \otimes 4tG_d) + \sum_{I: I \subset \{1, \dots, d-1\}, I \neq \emptyset} W_I^x(y_d | \alpha, 4t) \\ &\leq E\alpha(x_1 \otimes \dots \otimes x_{d-1} \otimes 4tG_d) + \sum_{I: I \subset \{1, \dots, d-1\}, I \neq \emptyset} W_I^x(y_d - x_d | \alpha, 4t) \\ &\quad + \sum_{I: I \subset \{1, \dots, d-1\}, I \neq \emptyset} W_I^x(\alpha, 4t) \\ &\leq E\alpha(x_1 \otimes \dots \otimes x_{d-1} \otimes 4tG_d) + \sum_{I: I \subset \{1, \dots, d\}, d \in I, I \cap \{1, \dots, d-1\} \neq \emptyset} W_I^x(\alpha, 4t) \\ &\quad + \sum_{I: I \subset \{1, \dots, d-1\}, I \neq \emptyset} W_I^x(\alpha, 4t) = \sum_{I: I \subset \{1, \dots, d\}, I \neq \emptyset} W_I^x(\alpha, 4t). \end{aligned}$$

The first inequality of this series of inequalities holds because of relation (A1). The second inequality was based on the first relation in the definition of the set  $A$  and on the definition of the set  $B$ . The third inequality is valid because of the relation  $\alpha(z \otimes y) \leq \alpha(z \otimes x) + \alpha(z \otimes (y - x))$  for arbitrary pseudonorm  $\alpha$  on the tensor product  $R^{n_1} \otimes \dots \otimes R^{n_d}$  and  $z \in R^{n_1} \otimes \dots \otimes R^{n_{d-1}}$ . The last inequality follows from the second relation in the definition of the set  $A$ . In the closing step we have applied the identity  $E\alpha(x_1 \otimes \dots \otimes x_{d-1} \otimes 4tG_d) = W_{\{d\}}^x(\alpha, 4t)$ .

Proposition 4.2 is a simple consequence of relations (A4) and (A5).

## References:

- 1.) Adamczak, R. (2006) Moment inequalities for  $U$ -statistics. *Annals of Probability* **34**, 2288–2314
- 2.) Latała, R. (2006) Estimates of moments and tails of Gaussian chaoses. *Annals of Probability* **34** 2315–2331
- 3.) Ledoux, M. (2001) The concentration of measure phenomenon. *Mathematical Surveys and Monographs* **89** American Mathematical Society, Providence, RI.
- 4.) Major, P. (2006) An estimate on the maximum of a nice class of stochastic integrals. *Probability Theory and Related Fields*. **134**, 489–537
- 5.) Major, P. (2007) On a multivariate version of Bernstein’s inequality *Electronic Journal of Probability* **12** 966–988